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STAGES OF MODELING AND SIMULATION, THROUGH PARTIAL DIFFERENTIAL EQUATIONS, FOR OVERAMORTIZED PROCESSES

Tiberiu COLOSI, Eva – Henrietta DULF Mihaela UNGURESAN
Mihai ABRUDEAN Silviu FOLEA Ioan NASCU

Technical University of Cluj-Napoca
Department of Automatic Control
Baritiu str. 26-28, Cluj-Napoca
e-mail: Tiberiu.Colosi@aut.utcluj.ro

Abstract

The present paper present the significant stages, which assure analogically modeling of processes, defined through second order partial differential equations (pde). The work enlarges a simple case study, considered enough representatively, for support extension to complex categories of this type of processes.

In paper are operated with step responses in exponential form, factorized form, with boundary conditions and initial conditions knows. At this base, are described an algorithm for calculus of coefficients for a pde, considered to be a typical case for a larges categories of overamortized processes.

An example executed on a computer details interesting conclusions in this context.

Keywords
partial differential equations, step response, initial and boundary conditions, algorithm of start, analogical modeling, numerical simulation

1. INTRODUCTION

Numerous technical and scientific processes (mechanic, electro technique, thermo technique, chemistry) are defined or approximated through pde by form:

$$P_{00}\cdot y + P_{10}\cdot\frac{\partial y}{\partial a} + P_{01}\cdot\frac{\partial y}{\partial b} + P_{20}\cdot\frac{\partial^2 y}{\partial a^2} + P_{11}\cdot\frac{\partial^2 y}{\partial a \partial b} + P_{02}\cdot\frac{\partial^2 y}{\partial b^2} = K_u\cdot u, \qquad (1)$$

for which the step response y = y(a,b) it consider to be of factorizing type, respectively:

$$y(a,b) = y_{00} + y_A(a)\cdot y_B(b)\cdot K_u\cdot u, \qquad (2)$$

Where (K_u) is proportionality coefficient and the input signal (u) are considered constant.

All the (a,b), $y_A(a)$ and $y_B(b)$ fulfills the continuity conditions (in Cauchy sense), and:

$$y_{00} = y(a_0,b_0), \qquad (3)$$

where $a_0 = 0$ and $b_0 = 0$.

The coefficients $P_{...}$ are constants and pde (1) can be of hyperbolic type ($P_{11}^2 - P_{20}\cdot P_{02} > 0$), of parabolic type ($P_{11}^2 - P_{20}\cdot P_{02} = 0$) or elliptic type ($P_{11}^2 - P_{20}\cdot P_{02} < 0$).

2. ESTABLISHING STEP RESPONSE, IN EXPONENTIAL FORM, FROM INITIAL AND BOUNDARY CONDITIONS

Is well known that the solutions of pde that models numerous technical-scientific processes, haves exponential functions which, in simplified form associated with (2), can be expressed in form:

$$y_A(a)=1 - \frac{T_{1a}}{T_{1a}-T_{2a}}\cdot\varepsilon^{-a/T_{1a}} - \frac{T_{2a}}{T_{2a}-T_{1a}}\cdot\varepsilon^{-a/T_{2a}} = 1 + \frac{1}{\lambda_a-1}\cdot\left(\varepsilon^{-\frac{\lambda_a\cdot\ln\lambda_a}{(\lambda_a-1)\cdot a_i}\cdot a} - \lambda_a\cdot\varepsilon^{-\frac{\ln\lambda_a}{(\lambda_a-1)\cdot a_i}\cdot a}\right), \quad (4)$$

respectively

$$y_B(b)=1 - \frac{T_{1b}}{T_{1b}-T_{2b}}\cdot\varepsilon^{-b/T_{1b}} - \frac{T_{2b}}{T_{2b}-T_{1b}}\cdot\varepsilon^{-b/T_{2b}} = 1 + \frac{1}{\lambda_b-1}\cdot\left(\varepsilon^{-\frac{\lambda_b\cdot\ln\lambda_b}{(\lambda_b-1)\cdot b_i}\cdot b} - \lambda_b\cdot\varepsilon^{-\frac{\ln\lambda_b}{(\lambda_b-1)\cdot b_i}\cdot b}\right), \quad (5)$$

The notations T_{1a}, T_{2a} and T_{1b}, T_{2b} it correspond to *time constants* at (a) direction, and (b) direction, respectively.

If (a) variable is considered time, and (b) variable represent a length, then naming of time constants (for T_{1a} and T_{2a}), respectively naming of length constants (for T_{1b} and T_{2b}) become rigorous naming. Is also noted $\lambda_a = T_{2a}/T_{1a} > 1$ and $\lambda_b = T_{2b}/T_{1b} > 1$, and ($a_i$) and ($b_i$) is inflexion abscises, such that are exemplified in Fig. 1.

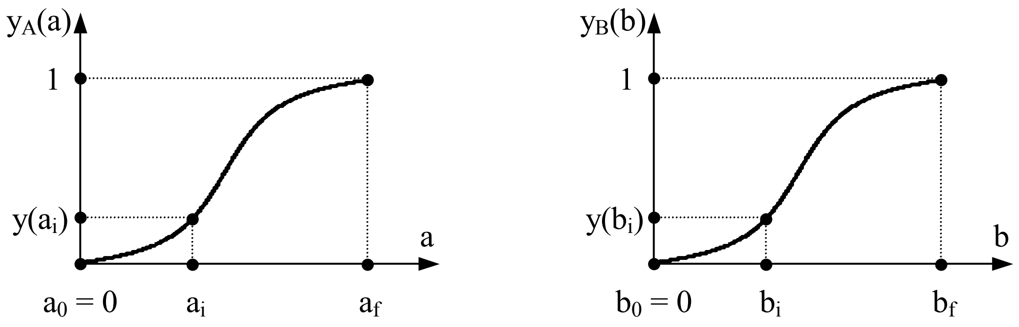

Fig. 1

From (4) and (5) it result that

$$y_A(0) = 0, \left(\frac{dy_A}{da}\right)_{a=0} = 0, \; y_A(\infty) = 1, \left(\frac{dy_A}{da}\right)_{a=\infty} = 0,$$

respectively

$$y_B(0) = 0, \left(\frac{dy_B}{db}\right)_{b=0} = 0, \; y_B(\infty) = 1, \left(\frac{dy_B}{db}\right)_{b=\infty} = 0,$$

Solving of transcendental equation (4), respectively

$$y_A(a) - 1 - \frac{1}{\lambda_a-1}\cdot\left(\varepsilon^{-\frac{\lambda_a\cdot\ln\lambda_a}{(\lambda_a-1)\cdot a_i}\cdot a} - \lambda_a\cdot\varepsilon^{-\frac{\ln\lambda_a}{(\lambda_a-1)\cdot a_i}\cdot a}\right) = 0, \quad (6)$$

can be make easiest on numerical way, through incrementing progressively of (λ_a) with step ($\Delta\lambda_a$) small enough.

Based on (λ_a) from (6), using inflexion condition

$$\left(\frac{d^2 y_A}{da^2}\right)_{a=a_i} = 0,$$

is calculated:

$$T_{1a} = \frac{\lambda_a - 1}{\lambda_a} \cdot \frac{a_i}{\ln \lambda_a} \text{ and } T_{2a} = \lambda_a \cdot T_{1a}. \qquad (7 \text{ and } 8)$$

Formally identical is acting for solving of transcendental equation (5), then result finally λ_b, respectively:

$$T_{1b} = \frac{\lambda_b - 1}{\lambda_b} \cdot \frac{b_i}{\ln \lambda_b} \text{ and } T_{2b} = \lambda_b \cdot T_{1b}. \qquad (7 \text{ and } 8)$$

With results thus obtained, are considered approximated (through variants (4) and (5)) the factorized form of step response (2). That follows, knowing of (a_i), (a_f), y(a_f), respectively (b_i), (b_f), y(b_f), then result finally (λ_a), (T_{1a}), (T_{2a}) respectively (λ_b), (T_{1b}) and (T_{2b}).

With referring at (2), can be observe that $y_{IC} = y(a_0,b)$ it correspond to initial conditions by time, usually rigorous knows, and $y_{FC2} = y(a_f,b)$ represent final conditions by time also, harder measurable, but theoretical appreciable. The boundary conditions at input, noted with $y_{FC1} = y(a,b_0)$, can be rigorous knows, and the output boundary

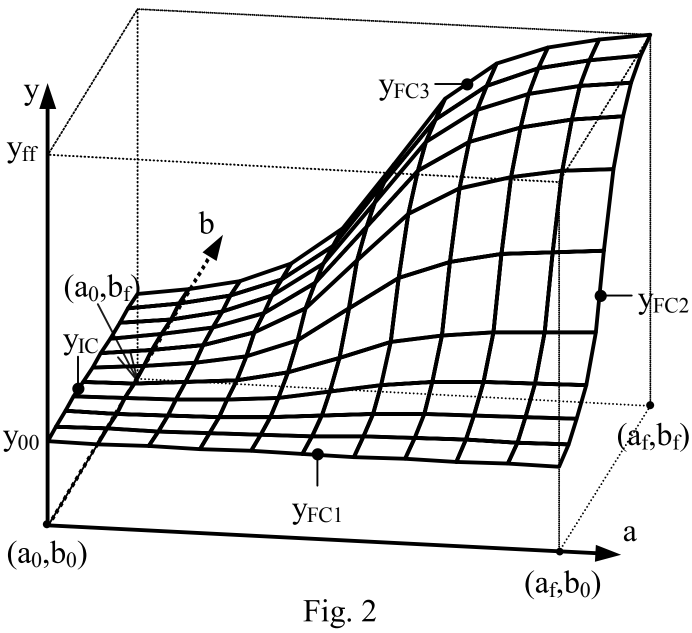

Fig. 2

conditions $y_{FC3} = y(a,b_f)$ can be fitted experimentally, with usually enough precision. These initial conditions, final conditions and boundary conditions are qualitative exemplified in Fig. 2.

3. THE CALCULUS OF THE COEFFICIENTS P…

After the experimentally and theoretically establishing of step response (2), with details specified in (3-10), it follow the calculus of coefficients $P_{...}$ that enter in compose of pde (1), which is consider to models these categories of processes.

Follow this scope, chousing $P_{00} = 1$, for calculating of the others coefficients (K_u, P_{10}, P_{01}, P_{20}, P_{11}, and P_{02}), must rewrite pde (1), in following six points (a_∞,b_∞), (a_0,b_f), (a_f,b_0), (a_f,b_∞), (a_∞,b_f) and (a_f,b_f), for u = 1 such following.

The stationary point reported at both abscissas ($a = \infty$; $b = \infty$), that for the stationery value $y_{st} = y(\infty,\infty)$ are considered known. Then result:

$$y_{st} = y(\infty,\infty) = y_{00} + y_A(\infty)\cdot y_B(\infty)\cdot K_u,$$

where $y_A(\infty) = 1$ and $y_B(\infty) = 1$, and

$$K_u = y_{st} - y_{00} \qquad (11)$$

The point (a_0,b_f), where $y(a_0,b_f) = y_{00}$, and all other differentials are null, excepting:

$$\left(\frac{\partial^2 y}{\partial a^2}\right)_{(a_0,b_f)} = \overset{2a}{y}_{0f} = \frac{1}{T_{1a}\cdot T_{2a}}\cdot y_B(b_f)\cdot K_u \qquad (12)$$

With these results, pde (1) become:

$$P_{00}\cdot y_{00} + P_{20}\cdot \overset{2a}{y}_{0f} = K_u \qquad (13)$$

The point (a_f,b_0), where $y(a_f,b_0) = y_{00}$, and all other differentials are null, excepting:

$$\left(\frac{\partial^2 y}{\partial b^2}\right)_{(a_f,b_0)} = \overset{2b}{y}_{f0} = \frac{1}{T_{1b}\cdot T_{2b}}\cdot y_A(a_f)\cdot K_u \qquad (14)$$

With these results, pde (1) become:

$$P_{00}\cdot y_{00} + P_{02}\cdot \overset{2b}{y}_{f0} = K_u \qquad (15)$$

The stationary point reported at abscise b ($a = a_f$; $b = \infty$), that for all differentials are null, excepting:

$$\left(\frac{\partial y}{\partial a}\right)_{(a_f,b_\infty)} = \overset{a}{y}_{f\infty} = \frac{1}{T_{1a}-T_{2a}}\cdot(\varepsilon^{-a_f/T_{1a}} - \varepsilon^{-a_f/T_{2a}})\cdot y_B(\infty)\cdot K_u, \qquad (16)$$

$$\left(\frac{\partial^2 y}{\partial a^2}\right)_{(a_f,b_\infty)} = \overset{2a}{y}_{f\infty} = \frac{1}{T_{1a}-T_{2a}}\cdot\left(-\frac{1}{T_{1a}}\cdot\varepsilon^{-a_f/T_{1a}} + \frac{1}{T_{2a}}\cdot\varepsilon^{-a_f/T_{2a}}\right)\cdot y_B(\infty)\cdot K_u, \qquad (17)$$

and $y(a_f,\infty) = y_{f\infty}$ is known value. With these results, pde (1) become:

$$P_{00}\cdot y_{f\infty} + P_{10}\cdot \overset{a}{y}_{f\infty} + P_{20}\cdot \overset{2a}{y}_{f\infty} = K_u. \qquad (18)$$

The stationary point reported at a abscise ($a = \infty$, $b = b_f$), which for all differentials are null, excepting:

$$\left(\frac{\partial y}{\partial b}\right)_{(a_\infty,b_f)} = \overset{b}{y}_{\infty f} = \frac{1}{T_{1b}-T_{2b}}\cdot(\varepsilon^{-b_f/T_{1b}} - \varepsilon^{-b_f/T_{2b}})\cdot y_A(\infty)\cdot K_u, \qquad (19)$$

$$\left(\frac{\partial^2 y}{\partial b^2}\right)_{(a_\infty,b_f)} = \overset{2b}{y}_{\infty f} = \frac{1}{T_{1b}-T_{2b}}\cdot\left(-\frac{1}{T_{1b}}\cdot\varepsilon^{-b_f/T_{1b}} + \frac{1}{T_{2b}}\cdot\varepsilon^{-b_f/T_{2b}}\right)\cdot y_A(\infty)\cdot K_u, \qquad (20)$$

and $y(\infty,b_f) = y_{\infty f}$ is known value. With these results, pde (1) become:

$$P_{00}\cdot y_{\infty f} + P_{01}\cdot y_{\infty f}^{b} + P_{02}\cdot y_{\infty f}^{2b} = K_u. \tag{21}$$

The final point (a_f,b_f), posses all values no null, such that pde (1) become:

$$P_{00}\cdot y_{ff} + P_{10}\cdot y_{ff}^{a} + P_{01}\cdot y_{ff}^{b} + P_{20}\cdot y_{ff}^{2a} + P_{11}\cdot y_{ff}^{ab} + P_{02}\cdot y_{ff}^{2b} = K_u\cdot u. \tag{22}$$

From system of five equations, respectively (13-22), it result all five coefficients, for $P_{00} = 1$, in this form:

$$P_{20} = \frac{K_u \cdot u - P_{00}\cdot y_{00}}{y_{0f}^{2a}}, \tag{23}$$

$$P_{02} = \frac{K_u \cdot u - P_{00}\cdot y_{00}}{y_{f0}^{2b}}, \tag{24}$$

$$P_{10} = \frac{K_u \cdot u - P_{00}\cdot y_{f\infty} - P_{20}\cdot y_{f\infty}^{2a}}{y_{f\infty}^{a}}, \tag{25}$$

$$P_{01} = \frac{K_u \cdot u - P_{00}\cdot y_{\infty f} - P_{02}\cdot y_{\infty f}^{2b}}{y_{\infty f}^{b}}, \tag{26}$$

$$P_{11} = \frac{K_u \cdot u - (P_{00}\cdot y_{ff} + P_{10}\cdot y_{ff}^{a} + P_{01}\cdot y_{ff}^{b} + P_{20}\cdot y_{ff}^{2a} + P_{02}\cdot y_{ff}^{2b}}{y_{ff}^{ab}}. \tag{27}$$

With these expressions is ending the calculus of coefficients $P_{...}$ from pde (1).

4. EXAMPLE, FOR A CASE STUDY

It is considered a step response, typical to thermal and chemical processes, formal exemplified in Fig. 2, for $a_f = 10$ h, $b_f = 6$ m, having the abscissas of inflexion points $a_i = 1.2$ h, respectively $b_i = 0.9$ m.

For $y_A(a)$ and $y_B(b)$ are approximated the expressions (4) and (5) respectively, which for $y_A(a_f) = 0.98$ m and $y_A(\infty) = 1$. The experimental step response (u = 1) was conducted at $y_{00} = 2$ and $y_{ff} = y(a_f,b_f) = 100$, having stationary component $y_{st} = y(\infty,\infty) = 104.4$. As follow, based on (2) and (11), it results $K_u = y_{st} - y_{00} = 102.4$.

For the calculus of T_{1a} and T_{2a} constants from (4) and T_{1b} and T_{2b} from (5), was built the program YINFL1(2), that assure transcendental equation (6) solving, then result the solution (λ_a), after that, such in (7) and (8), was obtain T_{1a} and T_{2a}. Formally identical, from (9) and (10) result T_{1b} and T_{2b}. Searched dates are:

a) $a_f = 10$ h; $a_i = 1.2$ h; $\lambda_a = 3.309998$; $T_{1a} = 0.699647$ h; $T_{2a} = 2.315889$ h;

$$y_A(a_f) = 0.98;\ y_A(a_i) = 0.2244399 \text{ and } \left(\frac{dy_A}{da}\right)_{a_i} = 0.2571867.$$

a) $b_f = 6$ m; $b_i = 0.9$ m; $\lambda_b = 1.969999$; $T_{1b} = 0.653773$ m; $T_{2b} = 1.287547$ m;

$$y_B(b_f) = 0.98;\ y_A(a_i) = 0.2505946 \text{ and } \left(\frac{dy_B}{db}\right)_{b_i} = 0.38606778.$$

Based on formulas (23-27), was designed the program COEF1(2); this are used with coefficients T_{1a}, T_{2a}, T_{1b} and T_{2b} above obtained, then was result: $P_{00} = 1$; P_{10}

0.6380584; P_{01} = 0.6142673; P_{20} = 1.619561; P_{11} = 157.7582; P_{02} = 0.8413306; K_u = 102.0408163265.

As follow, for this example, was finally obtain, pde (1) in complete form, that include establishing of initial conditions (IC), boundary conditions (FC), of structure parameters (T_{1a}, T_{2a}, T_{1b}, T_{2b}), and also of coefficients $P_{...}$, from (1).

5. CONCLUSIONS

1) Stages of analogical modeling and numerical simulation of, resuming and simplified exposed in this work, represent a case study, based on pde (1), with numerous applications in simulating and modeling of distributed parameters processes. This pde (1) can be considered a acceptable compromise, between complexity of phenomena and relative simplicity of mathematical formalism needed.
2) The approximation of factorizing step response, in form (2), was conduct to a good accuracy of phenomena interpretation, which for the exponential forms included in $y_A(a)$ and $y_B(b)$ from (4) and (5) respectively are justified both theoretically and experimentally.
3) Numerical solving of transcendental equation (6) reported to (λ_a), through the YINFL1(2) program is easy. It is based on theoretical or experimental knowing of abscises (a_f) and (a_i), and also of coordinate $y(a_f)$, then result time constants (T_{1a},T_{2a}). Formally identical it is obtained also the length constants (T_{1b},T_{2b}), all with remarkable phenomenological interpretation.
4) The program build, COEF1(2), from relations (23-27), and also at previous results, allow the calculus of coefficients from pde (1) composition.
5) Synthesizing from above, was started from a model of usual step response, formal exemplified in Fig. 2, and after walking of some stages, for analogical modeling and numerical simulation, was throughput the complete form of pde (1), including the coefficients $P_{...}$ and K_u. It also can be observed the remarkable possibilities to extend these stages of study also and for processes with distributed parameters reported at time and at a lot of spatial coordinates.

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