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THE TAYLOR – LIL METHOD FOR ANALOGICAL MODELLING AND NUMERICAL SIMULATION FOR LINEAR PROCESSES

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Abstract : Following the [1], [2], [3] works, and others, it is presented a possible approach to analogical modeling and numerical simulation, for linear, time invariant processes with concentrated parameters. The method called Taylor – LIL is presented in the complete version, and also with odd derivatives, respectively even derivatives. The advantages, restrictions and precautions that this method imposes are commented.

Keywords: state variables, Taylor series, local – iterative linearization (LIL), numerical integration, forced solutions, free solutions.

1. INTRODUCTION

This cathegory of processes is defined by the well-known vector – matrix form,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{2}$$

where (**u**), (**x**) and (**y**) are the input, state, output vectors, and (**A**), (**B**), (**C**), (**D**) correspond to the state, input – state, output – state, and input – output matrices.

These matrices are constant, the initial conditions (IC) at $t = t_0$, respectively $\mathbf{x}_{IC} = \mathbf{x}(t_0)$ are considered known. Given the usual hypothesis that the vector $u = u(t)$ has a continuous evolution relative to the time (t), the solutions for the ordinary differential equation (ode), in the vectorial form, respect the conditions of continuity in the Cauchy sense.

Proceeding to repeated derivatives (m-times), with respect to the time for (1), also

$$\overset{(m)}{\mathbf{x}} = A\overset{(m-1)}{\mathbf{x}} + \mathbf{B}\overset{(m-1)}{\mathbf{u}}, \tag{3}$$

where $m = 1, 2, 3, \ldots, \omega$, it is observed that this successive derivative of the state vector is greatly simplified, by using the lower order derivatives, respectively (m-1), (m-2), to m=1, determined in the equations obtained in the previous stages of derivation.

It is reminded that, relative to the same pivot moment $t_{k-1} = (k-1)\cdot\Delta t$, the Taylor series can be developed in a progressive or regressive variant. For example

$$\mathbf{x}_k \cong \mathbf{x}_{k-1} + \sum_{m=1}^{\omega} \frac{\Delta t^m}{m!} \cdot \overset{(m)}{\mathbf{x}}_{k-1} \tag{4}$$

respectively,

$$\mathbf{x}_{k-2} \cong \mathbf{x}_{k-1} + \sum_{m=1}^{\omega} (-1)^m \cdot \frac{\Delta t^m}{m!} \cdot \overset{(m)}{\mathbf{x}}_{k-1} \tag{5}$$

where $\mathbf{x}_{k-j} = \mathbf{x}(t_{k-j})$, for $t_{k-j} = (k-j)\cdot\Delta t$, and the regressive sequence $j \le k$. With $\Delta(t)$ we denote the integration step (small enough), which for usual applications is $\omega \le 6$.

Expression (4), containing odd and even derivatives, can be defined as “The Taylor series in the complete form”.

If (5) is substracted from (4) we obtain

$$\mathbf{x}_k \cong \mathbf{x}_{k-2} + 2 \cdot \left(\frac{\Delta t}{1!} \cdot \dot{\mathbf{x}}_{k-1} + \frac{\Delta t^3}{3!} \cdot \dddot{\mathbf{x}}_{k-1} + \ldots\right), \tag{6}$$

which represents “The Taylor series with odd derivatives”. By the lack of even derivatives a greater processing speed can be achieved, and sometimes a lower approximation error of the state vector (x_k).

If (5) is added to 4 we obtain

$$\mathbf{x}_k \cong 2 \cdot \mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2 \cdot \left(\frac{\Delta t^2}{2!} \cdot \ddot{\mathbf{x}}_{k-1} + \frac{\Delta t^4}{4!} \cdot \ddddot{\mathbf{x}}_{k-1} + \ldots\right), \tag{7}$$

which represents “The Taylor series with even derivatives”. By the lack of odd derivatives again a greater processing speed can be achieved, and sometimes a lower approximation error of the state vector (x_k).

It will also be used, the obvious relation

$$\mathbf{u}_{k-1} = \mathbf{u}_k + \sum_{m=1}^{\omega} (-1)^m \cdot \frac{\Delta t^m}{m!} \cdot \overset{(m)}{\mathbf{u}}_k, \tag{8}$$

necessary for substituting the sequence (k-1) of the input vector, with the (k) sequence, and also the relation (3), considered at the sequence (k-1), respectively

$$\overset{(m)}{\mathbf{x}}_{k-1} = \mathbf{A} \cdot \overset{(m-1)}{\mathbf{x}}_{k-1} + \mathbf{B} \cdot \overset{(m-1)}{\mathbf{u}}_{k-1}, \tag{9}$$

2. THE COMPLETE TAYLOR – LIL METHOD

The purpose is to approximate the state vector ($\mathbf{x}_k$), operating with (4), in which we substitute (9), resulting

$$\mathbf{x}_k \cong \mathbf{x}_{k-1} + \sum_{m=1}^{\omega} \frac{\Delta t^m}{m!} \cdot (\mathbf{A}\overset{(m-1)}{\mathbf{x}}_{k-1} + \mathbf{B}\overset{(m-1)}{\mathbf{u}}_{k-1}) \tag{10}$$

If for (10) we have m=1, the expression $(\mathbf{u}_{k-1)})$ is substituted with (8), after the resulting computation, (10) becomes

$$\mathbf{x}_k \cong \mathbf{g} \cdot \mathbf{u}_k + \mathbf{h}_k, \tag{11}$$

where

$$\mathbf{g} = \Delta t \cdot \mathbf{B} \tag{12}$$

and

$$\mathbf{h}_k = \mathbf{x}_{k-1} + \mathbf{A} \cdot \sum_{m=1}^{\omega} \frac{\Delta t^m}{m!} \cdot \overset{(m-1)}{\mathbf{x}}_{k-1} + \mathbf{B} \sum_{m=2}^{\omega} \frac{\Delta t^m}{(m-1)!} \cdot [\frac{1}{m} \cdot \overset{(m-1)}{\mathbf{u}}_{k-1} + (-1)^{m-1} \cdot \overset{(m-1)}{\mathbf{u}}_k]. \tag{13}$$

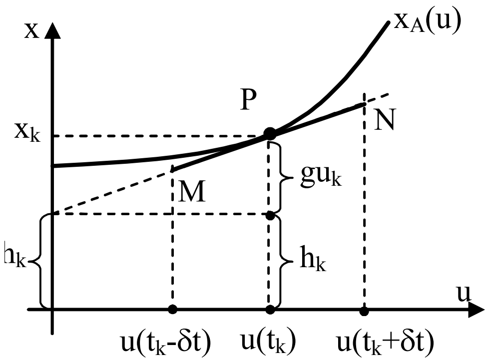

Figure 1

It can be observed that each element of vector (11), represents the equation of a line, exemplified in the scalar form in fig.1. It corresponds to the straight line segment ($\overline{MN}$), tangent in (P) with the analytical solution $x_A(u)$, from the immediate neighborhood of the input signal $u_k = u(t_k)$, inside the limits of the time interval $2 \cdot \delta t = \Delta t$, respectively $u(t_k - \delta t)$ and $u(t_k + \delta t)$. The interval (Δt), considered small enough is the integration step.

For a number of (n) state variables respectively $\mathbf{x}(n \times 1)$, there exists a family of (n) such lines , from the component of (11), with the remark that all (p) elements of the input vector $\mathbf{u}(p \times 1)$ are known.

As a result, the vector ($\mathbf{g}\mathbf{u}_k$) represents the forced component of vector ($\mathbf{x}_k$) from (11), and vector ($\mathbf{h}_k$) corresponds to the free component of the same vector. This free component ($\mathbf{h}_k$) from (11), contains all the regressive sequences from ($\overset{(m-1)}{\mathbf{x}}_{k-1}$), for m = 1, 2, …, ω, respectively from ($\overset{(m-1)}{\mathbf{u}}_{k-1}$) for m = 2, 3, …, ω, to which all the current sequences from ($\overset{(m-1)}{\mathbf{u}}_k$) are added, for m = 2, 3, …, ω, with the remark that all these sequences are known.

Because solution (11), corresponds to some families of lines, located in the immediate neighborhood of the pivot moment $t_k = k \cdot \Delta t$, respectively of the current sequence (k) which is incremented iteratively, it is just for (11) the denomination “ local – iterative linearization” (LIL) of the vector ($\mathbf{x}_k$).

The advantages of the formality (LIL) are presented in [1].

Because (11), (12) and (13) contain all even and odd order derivatives with respect to time (t), it is justified that the name of the method is Complete Taylor LIL.

3. TAYLOR-LIL METHOD WITH ODD DERIVATIVES

It pursues the approximation of state vector ($\mathbf{x}_k$) by operating with (6) in which (3) is inserted for m=1,3,5,…ω, resulting

$$\mathbf{x}_k \cong \mathbf{x}_{k-2} + 2\sum_{m=1}^{w}\left[\frac{\Delta t^{2m-1}}{(2m-1)!}\left(\mathbf{A}\overset{(2m-1)}{\mathbf{x}}_{k-1} + \mathbf{B}\overset{2m-1}{\mathbf{u}}_{k-1}\right)\right] \tag{14}$$

or

$$\mathbf{x} \cong \mathbf{x}_{k-2} + 2\mathrm{A}\left[\frac{\Delta t}{1!}\mathbf{x}_{k-1} + \frac{\Delta t^3}{3!}\ddot{\mathbf{x}}_{k-1} + \ldots\right] + 2\mathrm{B}\left[\frac{\Delta t}{1!}\mathbf{u}_{k-1} + \frac{\Delta t^3}{3!}\ddot{\mathbf{u}}_{k-1} + ..\right]. \tag{15}$$

If in (15) we insert (8) , in the end we obtain the same form as (11), for which

$$\mathbf{g} = 2\frac{\Delta t}{1!}\mathbf{B} \tag{16}$$

$$\mathbf{h}_k = \mathbf{x}_{k-2} + 2\mathbf{A}\sum_{m=1}^{w}\frac{\Delta t^{2m-1}}{(2m-1)!}\overset{(2m-2)}{\mathbf{x}}_{k-1} + 2\mathbf{B}\sum_{m=1}^{w}\left[\frac{\Delta t^{2m+1}}{(2m+1)!}\overset{(2m)}{\mathbf{u}}_{K-1} + (-1)^m\frac{\Delta t^m}{m!}\overset{(m)}{\mathbf{u}}_k\right] \tag{17}$$

And for this locally – iterative linearization form, the observations covered in chapter 2 are still valid.

4. TAYLOR -LIL METHOD WITH EVEN DERIVATIVES

It pursues the approximation of state vector ($\mathbf{x}_k$) by operating with (7) in which (3) is inserted for m=2,4,6,…ω, resulting

$$\mathbf{x}_k \cong 2\mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2\sum_{m=1}^{w}\left[\frac{\Delta t^{2m}}{(2m)!}\left(\mathbf{A}\overset{(2m)}{\mathbf{x}}_{k-1} + \mathbf{B}\overset{(2m)}{\mathbf{u}}_{k-1}\right)\right] \tag{18}$$

or

$$\mathbf{x}_k \cong 2\mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2\mathbf{A}\left[\frac{\Delta t^2}{2!}\dot{\mathbf{x}}_{k-1} + \frac{\Delta t^4}{4!}\ddot{\mathbf{x}}_{k-3} + \ldots\right] + 2\mathbf{B}\cdot\left[\frac{\Delta t^2}{2!}\dot{\mathbf{u}}_{k-1} + \frac{\Delta t^4}{4!}\ddot{\mathbf{u}}_{k-1} + \ldots\right]. \tag{19}$$

If, from Taylor series expansion of vector ($\dot{\mathbf{u}}_k$) with respect to pivot (k-1), it is expressed ($\dot{\mathbf{u}}_{k-1}$), we obtain

$$\dot{\mathbf{u}}_{k-1} \cong \frac{1}{\Delta t}\left[\mathbf{u}_k - \mathbf{u}_{k-1} - \sum_{m=2}^{w} \frac{\Delta t^m}{m!} \overset{(m)}{\mathbf{u}}_{k-1}\right]. \tag{20}$$

This result is inserted in (19), finally obtaining form (11) again, where (**g**) corresponds to (12), and

$$\begin{aligned}\mathbf{h}_k &= 2\mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2\mathbf{A}\cdot\sum_{m=1}^{\omega}\frac{\Delta t^{2m}}{(2m)!}\cdot \overset{(2m-1)}{\mathbf{x}}_{k-1} - \Delta t\cdot \mathbf{B}\mathbf{u}_{k-1} + \\ &+ \mathbf{B}\sum_{m=2}^{\omega}\left[2\frac{\Delta t^{2m}}{(2m)!}\cdot \overset{(2m-1)}{\mathbf{u}}_{k-1} - \frac{\Delta t^{m+1}}{m!}\cdot \overset{(m)}{\mathbf{u}}_{k-1}\right].\end{aligned} \tag{21}$$

For this locally-iterative linearized form, observations presented in chapter 2 are also valid.

5. CONCLUSIONS

The three numerical solutions for ($\mathbf{x}_k$), in complete variants, with odd and even derivatives, are brought to the same form (11), which inserted in (2), considered at sequence (k), leads to **Taylor-LIL numerical model**, for linear time invariant processes, with distributed parameters, defined by

$$\mathbf{x}_k = \mathbf{g}\mathbf{u}_k + \mathbf{h}_k \tag{11}$$
$$\mathbf{y}_k = (\mathbf{C}\mathbf{g} + \mathbf{D})\mathbf{u}_k + \mathbf{C}\mathbf{h}_k = \mathbf{g}_y\mathbf{u}_k + \mathbf{h}_{yk} \tag{22}$$

This vectorial equations system contributes to a unitary and systemized study of this type of processes, like, for example:

a) It cumulates the advantages of state variables use and of mathematical formalism resulted from (1) and (2).
b) By separating of forced components ($\mathbf{g}\mathbf{u}_k$ and $\mathbf{g}_y\mathbf{u}_k$) and free components ($\mathbf{h}_k$ and $\mathbf{h}_{yk}$) from (11) and (22), a simpler analytical interpretation of numerical solutions ($\mathbf{x}_k$) and ($\mathbf{y}_k$) is favorized.
c) Because input-state-output relations from (11) and (22), operate on the same current sequence (k), an algebra of base connection (series connection, parallel connection, feedback connection) can easily be elaborated, with remarkable advantages in the analysis and synthesis of complex systems.
d) Numerical approximation precision for ($\mathbf{x}_k$) is as much better as (Δt) is smaller and (ω) is bigger.
e) The Complete Taylor_LIL variant being a mono-step, the calculus start is always assured. After the first start step, we can skip, if this is the case, to the using of the other Taylor_LIL variants, with even or odd derivatives.

f) The logic scheme of numerical integration is simple, flexible and easily adaptable for a great diversity of processes.

g) In the case of signal discontinuities (from **u** vector's componence) or structure discontinuities (from **A**, **B**, **C**, **D** matrix's componence), a careful analysis of computation restarting is imposed, with a possible modification of (Δt) and (ω).

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