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CONTINUOUS AND DISCRETE SIGNAL SPACES ARE CONNECTED IN MORE INTERESTING WAYS

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Abstract: A general isometric isomorphism between the analog signal space L_2 and the digital signal space ℓ_2 which leave invariant the class of time-invariant realizable filters is presented. The isometric character of the isomorphism is proved based on a property of the functions mapping the imaginary axis on the unit circle. The associated orthonormal expansion is discussed. It is also shown how the previously known particular cases fit in the general solution.

Key words: isomorphism, isometric, L_2, ℓ_2, Hilbert space.

1. INTRODUCTION

Not all domains evolve equally. From time to time it make sense to try to find ways to move hard problems from one domain into another one, where more progress was made or problems have a chance to become (or look) simpler. The acceptable moves of the design between domains require preservation of specific rules known as invariants of the transformations. The chance that such methodology becomes more interesting grows when the transformations can be chosen from a larger class. Different quests for optimizations are open, and possibilities of understanding even more about the domains increase. Those were the motivating factors that lead to the generalization of the known [1] isometric isomorphism between the continuous and the discrete signal spaces that preserves the filter realizability, invariant required by both signal processing and pragmatic control.

There have been a couple extensions of the initial isomorphism (mapping) idea first introduced by K. Steiglitz [1]. Those were presented in [2], [3], and [6]. The aim was to develop classes of orthonormal functions for effective signal processing (transforms). For example, in order to get the required frequency resolution (closer to the human logarithmic scale), a modified Fourier kernel (complex exponentials) was proposed. If a uniform resolution spectrum is transformed by using this new warped basis (obtained by a proper inner function), and re-synthesized with a uniform (non-warped) basis, then a new spectrum of non-uniform resolution is created. The new frequency resolution depends on the actual inner function used in the warping. The orthogonalization was obtained by using the derivative of the inner function as a "weight". All the functions were normalized by the square root of this weight, in order to obtain orthonormalized bases. The topic of axis warping used to generate new function bases was analyzed in detail by R. G. Baraniuk and D. L. Jones [4]. A related concept named FAMlet was introduced in 1992 [5]. To the best of our knowledge no previous attempt was made to find the generalizations that preserve realizability.

The proposed mapping, being general, enlarges the exploration domain. It opens the door for optimizations, and the investigation of new methods for processing media signals or control signals. The hope is that with the dramatic increase in computation power, modern systems will afford more complex implementations, like those that become conceptually imaginable once a more general mapping between the analog and the discrete worlds is understood.

2. NOTATION

$\mathbb{N},\mathbb{Z},\mathbb{R}$ and $\mathbb{C}$ denote respectively the natural number set, the ring of rational integers, the real number field, and the complex number field. *, Re and Im mean complex conjugate, real part, and imaginary part (functions defined on $\mathbb{C}$).

3. BASIC PROPERTY

Let $(\mathcal{C}(U,U)\circ)$ denote the unit circle conformal group of all functions that map the disk $U=\{z\in\mathbb{C} \mid |z|<1\}$ onto itself conformally ($\circ$ denotes function composition). If h_0 is a function mapping the simply connected domain $D\subset\mathbb{C}$ conformally on U, then the set $\mathcal{C}(D,U)$ of all functions confromally mapping D on U, contains only the functions $f\circ h_0$, where $f\in\mathcal{C}(U,U)$. Particularly if D is the left-half complex plane $\mathrm{Re}(z)<0$ denoted by $\mathbb{C}_L$, choosing $h_0=(1+z)/(1-z)$, and knowing that $\mathcal{C}(U,U)$ is also the subgroup of linear-fractional functions of the form

$$z\mapsto f(z)=e^{jq}\frac{z-a}{1-zb*},(j=(0,1)\in\mathbb{C},q\in\mathbb{R},a\in U,b=h_0(a))$$

we find the most general function from $\mathcal{C}(\mathbb{C}_L,U)$

$$z\mapsto h(z)=e^{jf}\frac{1+zb}{1-zb*} \tag{3.1}$$

where

$$e^{jf}=e^{jq}\frac{1-a}{1-a*}$$

The set $\mathcal{C}(U,\mathbb{C}_L)$ contains the inverse mappings h^{-1}

$$z\mapsto h^{-1}(z)=\frac{z-e^{jf}}{zb*+e^{jf}b} \tag{3.2}$$

PROPOSITION 3.1. If $h^{-1} \in \mathcal{C}(U, \mathbb{C}_L)$ *and* $|\, z \,| = 1$, *then* $(\arg(h^{-1}{}'(z)) + \arg(z)) \in 2p\mathbb{Z}$

In other words, the argument of the derivative has the same magnitude, but opposite sign, as that of the argument of z, on the unit circle, for all functions mapping conformally the unit disk on the left half complex plane.

Proof. Using the notation

$$b + b^* = c^2, (b = h_0(a)) \tag{3.3}$$

$(c \in \mathbb{R}$ since $\mathrm{Re}(h_0(a)) > 0)$ the derivative of (3.2) is

$$(h^{-1}(z))' = e^{jf} c^2 \frac{1}{(zb^* + e^{jf} b)^2} \tag{3.4}$$

On the unit circle $z = e^{jx}$, with $x \in [0, 2p)$, and

$$\arg(h^{-1}(z))'_{z=e^{jx}} = \arg \frac{e^{jf}}{(e^{j(x-\arg(b))} + e^{j(f+\arg(b))})^2}$$

That gives

$$\arg(h^{-1}(z))'_{z=e^{jx}} = f - 2\frac{(x - \arg(b)) + (f + \arg(b))}{2} = -x$$

and this asserts the proposition. □

Choosing the positive constant c that satisfies (3.3), and designating by l the correspondence

$$z \mapsto l(z) = e^{j\frac{f}{2}} c \frac{1}{zb^* + e^{jf} b} \tag{3.5}$$

then

$$(h^{-1}(z))' = l^2(z) \tag{3.6}$$

and using proposition 3.1 we get

$$\frac{l(z) l^*(z)}{z} = l^2(z) \text{ if } |\, z \,| = 1 \tag{3.7}$$

Note that for every function in $\mathcal{C}(U, U)$, there is a corresponding function h in $\mathcal{C}(\mathbb{C}_L, U)$ (given by (3.1)), with inverse function h^{-1} in $\mathcal{C}(U, \mathbb{C}_L)$ (given by (3.2)) and the function l defined by (3.5) and satisfying (3.6) and (3.7). The interesting properties of the "l" function are useful during the demonstration of the isometric character of the new general isomorphism, further defined, between the continuous and the discrete signal spaces.

4. THE GENERAL ISOMORPHISM

The Hilbert Space (HS) $L_2(\mathbb{R})$ of complex valued, square integrable, Lebesgue measurable functions with the natural inner product

$$< f, g >_{L_2} = \int_{-\infty}^{+\infty} f(t) g^*(t) dt$$

and the HS ℓ_2 of double-ended, square summable, complex sequences with inner product

$$< x, y >_{\ell_2} = < \{x_n\}_{n\in\mathbb{Z}}, \{y_n\}_{n\in\mathbb{Z}} >_{\ell_2} = \sum_{n\in\mathbb{Z}} x_n y_n^*$$

are the spaces of continuous and discrete time signals respectively.
The space of Fourier transforms of functions in $L_2(\mathbb{R})$ designated by Φ is isomorphic as HS (i.e. isomorphic and isometric) with $L_2(\mathbb{R})$ since the Fourier transform $\mathcal{F}$ is an isomorphism and by Parseval's theorem, if $\mathcal{F}(f) = F_f$ and $\mathcal{F}(g) = F_g$, then

$$< f, g >_{L_2} = < F_f, F_g >_{\Phi} = \frac{1}{2p\, j} \int_{-j\infty}^{j\infty} F_f(s) F_g^*(s) ds \tag{4.1}$$

Also known is the HS isomorphism z (zeta) between the $\mathcal{Z}$ space of z-transforms of sequences in ℓ_2 and ℓ_2.
If the z-transform of $x \in \ell_2$ is $z(x) = F_x$ and $z(y) = F_y$, then by Parseval's relation

$$< x.y >_{\ell_2} = < F_x, F_y >_{\mathcal{Z}} = \frac{1}{2p\, j} \oint_{\partial U} F_x(z) F_y^*(z) \frac{1}{z} dz \tag{4.2}$$

where ∂U is the unit circle in the complex plane.
We define a general one to one correspondence

$$q : \Phi \to \mathcal{Z} \tag{4.3}$$

as follows

$$\Phi \ni f \xrightarrow{q} q(f) = l(f \circ h^{-1}) \in \mathcal{Z} \tag{4.4}$$

and

$$\mathcal{Z} \ni y \xrightarrow{q^{-1}} q^{-1}(y) = \frac{1}{l \circ h}(y \circ h) \in \Phi \tag{4.5}$$

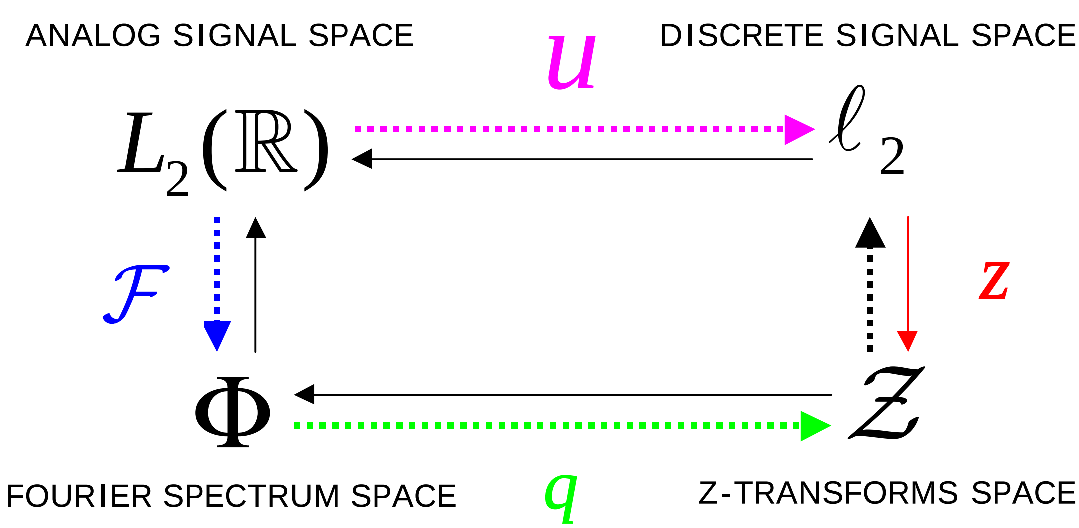

Figure 1 – Commutative function diagram of the four Hilbert space isometries.

It is now possible to construct an HS isomorphism between $L_2(\mathbb{R})$ and ℓ_2 which generalizes the one introduced by Steiglitz in [1]. We do that by defining u as $z^{-1} \circ q \circ \mathcal{F}$ (see figure 1). The isomorphic character of u is based on the fact that all transformations ($\mathcal{F}$, q and z) are one to one and linear. The proof of the isometric character of u is sketched below, using a series of equalities. Noted on top of each equal sign are the relevant implicants of respective equality.

$$< u(f), u(g) >_{\ell_2} \overset{(4.2)}{=} \frac{1}{2p\,j} \oint_{\partial U} q(F_f)(q(F_g))^* \frac{1}{z} dz \overset{(4.4)}{=} \tag{4.6}$$

$$\frac{1}{2p\,j} \oint_{\partial U} \frac{l(z) l^*(z)}{z} ((F_f \circ h^{-1})(F_g \circ h^{-1})^*)_z dz \overset{(3.6),(3.7)}{=} \tag{4.7}$$

$$\frac{1}{2p\,j} \oint_{\partial U} (h^{-1}(z))' F_f(h^{-1}(z)) F_g^*(h^{-1}(z)) dz \overset{h^{-1}(z)=w}{=} \tag{4.8}$$

$$\frac{1}{2p\,j} \int_{-j\infty}^{j\infty} F_f(w) F_g^*(w) dw = < f, g >_{L_2} \tag{4.9}$$

5. ORTHONORMAL FUNCTIONS ATTACHED

For every isomorphism u, we can find a set of orthonormal functions $\{l_n(t)\}_{n\in\mathbb{Z}}$ so that, if $f \in L_2(\mathbb{R})$ is mapped to $\{f_n\}_{n\in\mathbb{Z}} \in \ell_2$, then the equality $f_n = < f, l_n >_{L_2}$ holds for every $n \in \mathbb{Z}$. We will prove this by substituting $h(w)$ for z in

$$f_n = \frac{1}{2p\,j} \oint_{\partial U} (l(F_f \circ h^{-1}))_{(z)} z^n \frac{dz}{z}$$

getting

$$f_n = \frac{1}{2p\,j} \int_{-j\infty}^{j\infty} F_f(w) \left[(l \circ h) h' h^{n-1} \right]_{(w)} dw$$

$$f_n = \int_{-\infty}^{\infty} f(t)\, l_n^*(t) dt = < f, l_n >_{L_2}$$

The functions l_n are given by the following two-sided Laplace transforms:

$$l_n(t) = \mathcal{L}_B^{-1} \left[c_n \frac{\left(\frac{1}{b^*} - w \right)^{n-1}}{\left(\frac{1}{b} - w \right)^{n}} \right]$$

where

$$c_n = e^{j\frac{q}{2}(1-2n)} c \frac{(b^*)^{n-1}}{b^n}$$

After calculations, we find

$$l_n = \begin{cases} (-1)^{n-1} c_n e^{\frac{1}{b}} L_{n-1}\left[\left(\frac{1}{b}+\frac{1}{b^*}\right)t\right] s\,(t) \text{ for } n \in \mathbb{N}-\{0\} \\ (-1)^{n} c_n e^{\frac{1}{b^*}} L_{-n}\left[-t\left(\frac{1}{b}+\frac{1}{b^*}\right)\right] s\,(-t) \text{ for } n \in -\mathbb{N} \end{cases}$$

Where $L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right)$ is the Laguerre polynomial of degree n, and $s\,(t)$ is the Heaviside unit step function. A time invariant analog filter has an image in Φ, $\mathcal{Z}$ and ℓ_2 for every isomorphism u. The images A in Φ and $\mathcal{A}$ in $\mathcal{Z}$ are related by

$$A = \mathcal{A} \circ h \text{ and } \mathcal{A} = A \circ h^{-1}$$

and if A is realizable, $\mathcal{A}$ is also realizable and vice versa.

6. CONCLUSIONS

We can now use in applications not only the isomorphism introduced in [1] (which is a particular case for $a = 0$ and $q = 0$), but a family of isomorphisms indexed on $U \times \frac{\mathbb{R}}{2p}$ and every member of that family preserves the realizability of time invariant filters and the norm. Future investigations could answer whether any isometric isomorphism between the analog and the discrete signal spaces is of the form described in this paper. The wider space of optimizations opened by the generalization could be leveraged in DSP and control applications.

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