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AN EXAMPLE OF A TIME VARYING STATE SPACE DECOMPOSITION

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Abstract

State space decompositions are a fundamental tool in control system theory. The classical notion of state appears in connection with the usual differential system, where the state space is the same at every moment, simply because this system is time-invariant. Generalizations of this concept to abstract time-varying systems were develloped in [1], [2]. In this paper we apply this general theory to some particular cases and we present a new example of a system whose minimal state space decomposition is necessarly time-varying

Key words: *Time-varying state space, controllability, observability, causality.*

I. General results

1.Introduction

Let $L^2(0,1)$ be the usual Hilbert space of square-integrable signals on $[0,1]$, let $A \in \mathcal{M}_n$, $B \in \mathcal{M}_{n,1}$, $C \in \mathcal{M}_{1,n}$ be real matrices and let $\mathcal{D} : L^2(0,1) \mapsto L^2(0,1)$, $\mathcal{D}u = y$ be the usual differential system, i.e. $x'(t) = Ax(t) + Bu(t), x(0) = 0$, $y(t) = Cx(t)$. The state space of the system is R^n, the state of the system at every moment $t \in [0,1]$ is the vector $x(t)$. Further, classical system theory developed tests of controllability and observability in terms of the matrices A, B, C. In the following, we develop a general state space decomposition theory for an arbitrary system on a Hilbert space and generalize the usual observability and controlability tests for the differential system. An important property of this theory is the fact that the state space is time-varying, i.e the state space has different dimensions at different moments. In the second part, we apply these general results to two particular systems. One of them is a new example of a system which doesn't have a minimal time-invariant state decomposition.

Let H be a Hilbert space and let $\mathcal{T}$ be a totally ordered set with t_0 and t_∞ the minimum and maximum elements, respectively. A family $\mathcal{P} = (P_t)_{t\in\mathcal{T}}$ of orthogonal projections on H is called resolution if:

(a) $P_t \leq P_s, \forall t \leq s$.
(b) $P_{t_0} = 0$, $P_{t_\infty} = I$.
(c) If P_{t_n} tends strongly to P, then $P \in \mathcal{P}$.
The pair $(H, \mathcal{P})$ is called a Hilbert resolution space. If $\ell^2(N)$ is the Hilbert space of square integrable discrete signals, then the time set $\mathcal{T} = N \cup \infty$ and the projections are $(P_k x)(n) = x(n)$, $\forall n \leq k$ and $(P_k x)(n) = 0$, $\forall n > k$, $\forall x \in \ell^2(N)$, $\forall k \in N$. If $H = L^2(0,1)$, then the time set is $\mathcal{T} = [0,1]$ and the projections are
$(P_t u)(s) = u(s), \forall s \leq t$ and $(P_t u)(s) = 0, \forall s > t$, $\forall u \in L^2(0,1)$, $\forall t \in [0,1]$.
Let $\mathcal{L}(H)$ be the Banach algebra of all linear bounded systems on H and let $T \in \mathcal{L}(H)$. The system T is called causal if $P_t T = P_t T P_t$, $\forall t \in \mathcal{T}$, or, equivalently all the subspaces $\ker(P_t)$ are invariant for T.
We now give the definition of the main concept.
A state decomposition for the system $T \in \mathcal{L}(H)$ is defined by a a family
$(K_t, \alpha_t, \beta_t)_{t \in \mathcal{T}}$, such that:
(d) K_t is a Hilbert space, $\forall t \in \mathcal{T}$.
(e) $\alpha_t : H \mapsto K_t$ is a bounded linear system such that $\alpha_t = \alpha_t P_t$, $\forall t \in \mathcal{T}$.
(f) $\beta_t : K_t \mapsto H$ is bounded linear system such that $\beta_t = (I - P_t)\beta_t$, $\forall t \in \mathcal{T}$.
(g) $(I - P_t) T P_t = \beta_t \alpha_t$, $\forall t \in \mathcal{T}$.
Two state decompositions (K_t, α_t, β_t) and $(K'_t, \alpha'_t, \beta'_t)$ are said to be equivalent if there is a family of bounded invertible operators $A_t : K_t \mapsto K'_t$ such that $\alpha'_t = A_t \alpha_t$ and $\beta'_t = \beta_t A_t^{-1}$, $\forall t \in \mathcal{T}$. For details see [1],[3],[4],[5],[6].

2.Definitions

A state decomposition $(K_t, \alpha_t, \beta_t)_{t \in \mathcal{T}}$ is called controllable if
$\forall t \in \mathcal{T}$, $\forall x \in K_t \forall \epsilon > 0, \exists \delta_\epsilon > 0$ and $u \in H$ such that $\| \alpha_t u - x \| < \epsilon$.
The state decomposition is called observable if
$\forall t \in \mathcal{T}$, $\exists m > 0$ such that $\| \beta_t x \| \geq m \| x \|, \forall x \in K_t$.
The state decomposition is called minimal if it is controllable and observable.

3.Proposition

Let $(H, \mathcal{P})$ be a Hilbert resolution space and let $T \in \mathcal{L}(H)$. Then T admits a minimal state decomposition.
Proof For every $t \in \mathcal{T}$ we define the equivalence on $P_t(H)$ by

$$x \sim y \Leftrightarrow (I - P_t) T P_t x = (I - P_t) T P_t y.$$

The set of classes of equivalence, denoted by S_t can be organized as an inner space with the operations: $\hat{x} + \hat{y} = \widehat{x + y}$, $\lambda \hat{x} = \widehat{\lambda x}$, and the inner product:

$$< \hat{x}, \hat{y} >=< (I - P_t) T P_t x, (I - P_t) T P_t y >, \forall \hat{x}, \hat{y} \in S_t, \forall \lambda \in C.$$

Let K_t be the closure of S_t. The maps α_t and β_t are defined as follows: $\alpha_t u = \widehat{P_t u}$ and $\beta_t \hat{x} = (I - P_t) T P_t x$, $\forall t \in \mathcal{T}$. Checking that $(K_t, \alpha_t, \beta_t)_{t \in \mathcal{T}}$ is standard, ([1]).

The unicity of the minimal state decomposition is given by the following result (for details see [1],[2]).

4.Proposition

Every two minimal state decompositions of the same system are equivalent.
Proof Let $T \in \mathcal{L}(H)$ be a system and let (K_t, α_t, β_t) and $(K'_t, \alpha'_t, \beta'_t$ be two state space decompositions associated to T. We define $A_t : K_t \mapsto K'_t$ on the range of α_t

(which is dense in K_t). Let $u \in H$ and let $x = \alpha_t u \in \alpha_t(H)$. The operators A_t are defined by $A_t x = \alpha'_t u$. If $x = \alpha_t u = \alpha_t v$, then

$$\beta'_t \alpha'_t u = (I - P_t)TP_t u = \beta_t \alpha_t u = \beta_t \alpha_t v = (I - P_t)TP_t v = \beta'_t \alpha'_t v,$$

hence $\alpha'_t u = \alpha'_t v$, since β'_t is one to one. It results that the operators A_t are well defined. The relations $A_t \alpha_t = \alpha'_t$ and $\beta'_t A_t = \beta_t$ are obvious. We have to prove that the operators A_t are bounded and invertible. Let$x = \alpha_t u, u \in H$and let $m > 0$ such that $\| \beta_t y \| \geq m \| y \|, y \in K'_t$. Then:

$$\| A_t x \| \leq \frac{1}{m} \| \beta'_t A_t x \| = \frac{1}{m} \| \beta_t x \| \leq \frac{\| \beta_t \|}{m} \| x \|,$$

hence A_t is bounded. Analogously:

$$\| A_t x \| \geq \frac{m}{\| \beta'_t} \| x \|,$$

hence A_t is bounded above and below and has dense range; it results that A_t is invertible ([7]).

5. Observation

It can be proved ([1],[2]) that for a family $(K_t, \alpha_t, \beta_t)_{t \in \mathcal{T}}$ satisfying conditions d,e,f of the Introduction and it is minimal (in the sense of definition 2) there is (but it is not unique) a causal system whose the above family is a minimal state decomposition.

The above setup allows a general test for controllability and observability which generalizes the classical test for the differential system.

6.Theorem

Let $(H, \mathcal{P})_{t \in \mathcal{T}}$ be a Hilbert resolution space, let $T \in \mathcal{L}(H)$ and let (K_t, α_t, β_t) be a state decomposition for T.
(a) The decomposition is controllable if and only if $\alpha_t \alpha_t^\star > 0, \forall t \in \mathcal{T}$.
(b) The decomposition is observable if and only if $\beta_t^\star \beta_t > 0, \forall t \in \mathcal{T}$.
Proof (a) If the state decomposition is controllable, then, by definition, it results that the operators α_t have dense range. It results ([7], prop.29,ch.3) that the operators $\alpha_t^\star$ are one to one; it results that $\forall u \in H$:

$$< \alpha_t \alpha_t^\star u, u > = \| \alpha_t^\star u \|^2 > 0.$$

Conversely, if $\alpha_t \alpha_t^\star > 0$, then the operators $\alpha_t^\star$ are one to one, hence, by the same argument, the operators α_t have dense range. Analogously, one can prove (b).

II. Applications

We now apply the general results of the first section to some particular systems.

7.The discrete system

Let $A \in \mathcal{M}_n$, $B \in \mathcal{M}_{n,1}$, $C \in \mathcal{M}_{1,n}$; The discrete system $\mathcal{Z} : \ell^2(N) \mapsto \ell^2(N), \mathcal{Z}u = y$, where $y(k) = Cx(k), \forall k \in N$ and $x : N \mapsto R^n$ is the solution of the recurence $x(k+1) = Ax(k) + Bu(k), x(0) = 0$.
We shall define a state decomposition and we shall get a test of controllability and observability for this system as an application of the general test from theorem 6,

(for details see [3],[4]).
Let us first observe that

$$y(k) = Cx(k) = C\sum_{j=0}^{k-1} A^j Bu(k-1-j), \forall k \geq 1.$$

Let $\mathcal{T} = N \cup \infty$ be the time set and let P_m be the truncation operators $(P_m u)(k) = u(k)$, $\forall k \leq m$ and $(P_m u)(k) = 0$, $\forall k > m$. For every $m \in N$ we define the state space $K_m = R^n$. It results that the state space decomposition is time-invariant, since the state space is the same at every moment, $m \in N$. We now define the maps α and β as follows.
Let $k \in N$ and let

$$\alpha_k : \ell^2(N) \mapsto R^n, \alpha_k u = x(k) = \sum_{j=0}^{k-1} A^j Bu(k-1-j),$$

$$\beta_k : R^n \mapsto \ell^2(N), (\beta_k \gamma)(j) = \begin{cases} CA^{j-k}\gamma & , \quad j \geq k \\ 0 & , \quad j \leq k-1 \end{cases}$$

We now prove that this is a state space decomposition, i.e properties d,e,f,g from the Introduction.
Let $k \in N, u \in \ell^2(N)$ and $\gamma \in R^n$; then:

$$\alpha_k u = \sum_{j=0}^{k-1} A^j Bu(k-1-j) = \sum_{j=0}^{k-1} A^j B(P_k)u(k-1-j) = \alpha_k P_k u,$$

$$(\beta_k \gamma)(j) = \begin{cases} CA^{j-k}\gamma & , \quad j \geq k \\ 0 & , \quad j \leq k-1 \end{cases} = ((I - P_k)\beta_k \gamma)(j).$$

Moreover, if $j \leq k-1$, then:

$$((I - P_k)\mathcal{Z}P_k u)(j) = 0 = (\beta_k \alpha_k u(j).$$

If $j \geq k$, then:

$$(\beta_k \alpha_k u)(j) = (\beta_k x(k))(j) = CA^{j-k}x(k) = CA^{j-k}\sum_{m=0}^{k-1} A^m Bu(k-1-m) =$$

$$= \sum_{m=0}^{k-1} CA^{j-k+m}Bu(k-1-m) = ((I - P_k)\mathcal{Z}P_k u)(j).$$

It should be noticed that the above state decomposition is time-invariant since the state space is the same at every moment: R^n. The reason is that the discrete system is time-invariant.
We now prove that the usual test of observability and controllability for this system is a particular case of the general test from theorem 5:
(a) The above state decomposition is observable if and only if the matrix
$(C; CA; CA^2; ...; CA^{n-1})^T$ has the rank n.
(b) The state decomposition is controllable if and only if the matrix
$(B; BA; BA^2; ...; BA^{n-1})$ has the rank n.

We prove assertion (a). According to theorem 5, the state decomposition is observable if and only if $\beta_k^\star \beta_k > 0, \forall k \in N$; we first compute the adjoint $\beta_k^\star : \ell^2(N) \mapsto R^n$. For every $u \in \ell^2(N)$ and $\gamma \in R^n$ we get:

$$< u, \beta_k \gamma >= \sum_{j\geq 0} u(j)(\beta_k\gamma)(j) = \sum_{j\geq k} u(j) C A^{j-k}\gamma =$$

$$= \left(\sum_{j\geq k} (A^T)^{j-k} C^T u(j) \right)^T \gamma =< \sum_{j\geq k} (A^T)^{j-k} C^T u(j), \gamma >,$$

hence $\beta_k^\star u = \sum_{j\geq k} (A^T)^{j-k} C^T u(j), \forall u \in \ell^2(N)$.
It results that $\beta_k^\star \beta_k \gamma = \left(\sum_{j\geq k} (A^T)^{j-k} C^T C A^{j-k} \right) \gamma$. Hence the state decomposition is observable if and only if the matrix $\sum_{j\geq k} \left(CA^{j-k} \right)^T \left(CA^{j-k} \right)$ is positive, i.e.:

$$\sum_{j\geq k} \left(CA^{j-k}\gamma \right)^T \left(CA^{j-k}\gamma \right) > 0,\ \forall \gamma \in R^n, \gamma \neq 0.$$

It results that the decomposition is not observable iff there is a nonzero vector $\gamma \in R^n$ such that $CA^m\gamma = 0, \forall m \in N$; by applying the Hamilton-Caley theorem this is equivalent to $CA^m\gamma = 0, \forall m \in \{0, 1, ..., n-1\}$, or, equivalently, $\det \left(C; CA; CA^2; ...; CA^{n-1}\right)^T = 0$.

8.A time-varying minimal state decomposition

In the previous example the state space was time-invariant, i.e. it was the same at every moment. The general theory of the first section allows to define time-varying minimal state decompositions. We now introduce a new example of system for which a minimal state decomposition is necessarily time-varying.
Let $H = \ell^2(Z)$ and let $\{\sigma_n\}_{n\in Z}$ be its usual orthonormal basis, i.e. $\sigma_n(j) = \delta_{jn}$. For every moment $n \in Z$, let

$$K_n = R^{2|n|+1} = \{(x_{-n}, x_{-n+1}, ..., x_0, x_1, ..., x_n)\,;\, x_j \in R\}$$

be the state space at n and let

$$\alpha_n : \ell^2(Z) \mapsto R^{2|n|+1},\ \alpha_n u = (u(-n), u(-n+1), ..., u(0), ..., u(n))$$

$$\beta_n : R^{2|n|+1} \mapsto \ell^2(Z),\ \beta_n(x_{-n}, ..., x_0, ..., x_n) = \sum_{k=-n}^{n} x_k \sigma_{2n+k+1}.$$

Relations $\alpha_n P_n = \alpha_n$ and $(I - P_n)\beta_n = \beta_n$ are direct consequences of the definition. It results (Observation 5; for details see [2]) that $(R^{2|n|+1}, \alpha_n, \beta_n)_{n\in Z}$ is indeed a state decomposition for some system T on $\ell^2(Z)$ which must satisfy the equality:

$$(I - P_n)TP_n u = \beta_n \alpha_n u = \beta_n(u(-n), ..., u(0), ..., u(n)) =$$

$$= \sum_{k=-n}^{n} u(k)\sigma_{2n+k+1},\ \forall u \in \ell^2(Z).$$

Moreover, the decomposition is a time-varying state decomposition, since the state space is changing at every moment n: $R, R^3, R^5, ...$

We now prove that this state space decomposition is minimal.
If $n \in Z$, $x \in R^{2|n|+1}, u \in \ell^2(Z)$, then:

$$< x, \alpha_n u >= \sum_{k=-n}^{n} x_k u(k) = \sum_{k \in Z} (\alpha_n^{\star} x)(k) u(k),$$

where $\alpha_n^{\star} x = \sum_{k=-n}^{n} x_k \sigma_k$.
It results $\alpha_n \alpha_n^{\star} = I > 0$, hence, by applying theorem 6 it results that the decomposition is controllable.
Let $n \in Z, u \in \ell^2(Z), x \in R^{|n|+1}$; then:

$$< u, \beta_n x >= \sum_{k \in Z} u(k)(\beta_n x)(k) =$$

$$= u(n+1)x_{-n} + u(n+2)x_{-n+1} + ... + u(3n+1)x_n = \sum_{k=-n}^{n} x_k (\beta_n^{\star} u)(k),$$

where $\beta_n^{\star} u = (u(n+1), u(n+2), ..., u(3n+1))$.
Since $\beta_n^{\star} \beta_n = I > 0$, it results, by applying again theorem 6, that the state decomposition is observable, too, hence it is minimal. Obviously, there is no time-invariant minimal state space decomposition for such a system.

III. Conclusions

Time-invariant state space decompositions are useful only in the study of time-invariant systems. In order to define minimal state space decompositions for time-variant systems, the above general theory is a necessary tool, which can be applied to a large variety of systems, as the two previous examples show.

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