A CLASS OF DYNAMICAL SYSTEMS DISCONTINUOUS WITH RESPECT THE STATE VARIABLE

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Abstract: A class of dynamical systems discontinuous with respect the state variable is presented. To introduce this class, the initial value problem is transformed into a differential inclusion using the Filippov regularization. In order to integrate the multivalued initial value problem, the explicit Euler method for differential inclusions is used. Two significant examples are presented.

Keywords: Filippov regularization, differential inclusion, switch dynamical system.

1 INTRODUCTION

Let us consider the following initial value problem (i.v.p.) modeling a special class of discontinuous dynamical systems (d.s.)

$$\dot{x}(t) = f(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i \operatorname{sgn} x_i(t) e^i, \quad x(0) = x_0, \quad t \in I = [0, \infty), \qquad (1.1)$$

where $f, g: \mathbf{R}^n \to \mathbf{R}^n$ are single-valued vector functions, $\alpha_i \in \mathbf{R}$ and e^i is the i-th canonical unit vector in $\mathbf{R}^n$. The function f is piecewise continuous , i.e. continuous on a finite number of open domains $D_i \subset \mathbf{R}^n$, $i = 1,.., p$ in each of which being continuous up to the boundary, and having finite (possible different) limits from different boundary points (i.e. bounded discontinuities). The set of zero measure $M = \mathbf{R}^n \setminus \bigcup_{i=1}^{p} D_i$ contains the boundaries of D_i and represents the set of discontinuity points of f. The discontinuity (with respect to the state variable x), is due to the sign function

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}.$$

These classes of discontinuous d.s., we call switch, are less studied, due to the tedious work necessary to investigate the underlying i.v.p. Nevertheless, many physical laws are expressed by this kind of discontinuity and occur in many real problems as example the discontinuous dependence of the friction force on the velocity in the cases of dry friction, oscillating systems with combined dry and viscous damping, electrical circuits, forced vibrations, brake processes with locking phase, convex optimization, control synthesis, uncertain systems, etc. (see e.g. [1, 5, 16, 17, 20, 19]). Certain properties as the possibility to synchronize two chaotic switch d.s. [7], chaotification of switch d.s. having not chaotic behavior [8], continuous approximation of switch d.s. [9], numerical approximation using numerical methods for differential inclusions [6] were studied by the author.

2 SWITCH DYNAMICAL SYSTEMS

In order to find the assumptions on f under which an discontinuous i.v.p. as (1.1) defines a switch d.s. some notions and preliminary results will be present.

Definition 2.1. [6] The i.v.p. (1.1) is said to define a **switch dynamical system** if the i.v.p. admits at least a solution..

The classical notion of solution to differential equations is useless in the cases of discontinuous d.s. As example let us consider the following example [12]

$$\dot{x} = 1 - 2\,\mathrm{sgn}(x), \quad x(t_0) = 0, \quad t_0 > 0, \tag{2.1}$$

which for $x \neq 0$ has the following classical solutions (Figure 1)

$$x(t) = \begin{cases} 3t + C_1, & \text{for } x < 0 \\ -t + C_2, & \text{for } x > 0 \end{cases}.$$

As t increases, these solutions tend to the line $x = 0$, which cannot be taken as solution since the function $x(t) = 0$, does not satisfy the equation. Thus there is no classical solution starting from 0.

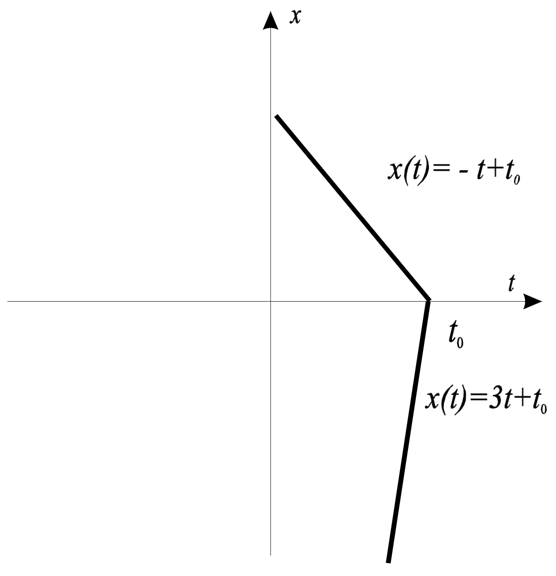

Figure 1.
Classical solutions to i.v.p. (2.1).

Filippov had the idea to replace the discontinuous differential equation with a differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad \text{for a.a. } t \in I, \tag{2.2}$$

where $F:\mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is a set-valued vector function on the set of all subsets of $\mathbf{R}^n$. The simplest convex definition of F is

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(M)=0} \overline{conv}(f(\{z \in \boldsymbol{R}^n : \| z - x \| \le \varepsilon\} \setminus M)), \tag{2.3}$$

where μ is the Lebesgue measure, M the set of the discontinuity points, $\overline{conv}$ is the closed convex hull and f an single-valued function discontinuous with respect the state variable. In the points where the function f is continuous, $F(x)$ consists of one point which coincides to the value of f at this points. In the discontinuity points, the set $F(x)$ is a subset of $\mathbf{R}^n$ being given by (2.3). In order to justify the use of the Filippov regularization in physical systems, ε must be small enough such that the motion of the physical system be arbitrarily close to a certain solution of the differential inclusion. The M set for the i.v.p. (1.1) is the set of points where the scalar functions $\operatorname{sgn} x_i$ vanish.
The set-valued version of the usual sign function is the sign set-valued Sgn function (Figure 2)

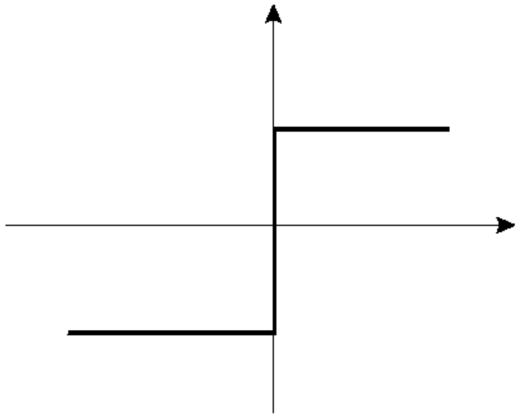

Figure 2.
The graph of the set-valued sign function.

$$\operatorname{Sgn} x = \begin{cases} \{-1\} & x < 0 \\ [-1,1] & x = 0 \\ \{+1\} & x > 0 \end{cases}.$$

Using Filippov regularization to the i.v.p. (1.1) becomes

$$\dot{x}(t) \in F(x(t)) := g(x(t)) + \sum_{i=1}^{n} \alpha_i \operatorname{Sgn} x_i(t) e^i, \quad x(0) = x_0, \text{ for a.a. } t \in I. \tag{2.4}$$

Now, we can define the notion of generalized solution to i.v.p. (1.1) via differential inclusions (see [6-12]).

Definition 2.2. [12]. A **generalized (Filippov) solution** to i.v.p. (1.1) is an absolutely continuous vector function $x: I \to \mathbf{R}^n$ satisfying (2.4) a.e. on I.

In others words, the solutions of the i.v.p. (2.4) are solutions to i.v.p. (1.1).

Let consider again as example the i.v.p. (2.1). Using the Filippov regularization, one obtain the following differential inclusion

$$\dot{x} \in 1 - 2\operatorname{Sgn}(x), \quad x(t_0) = 0, \quad \text{for a.a. } t \in I.$$

Hence, the generalized solutions to i.v.p. (2.1) are: one positive $x(t) = -t + t_0$ for $t < t_0$, and $x(t) = 0$ for $t \ge t_0$ and one negative $x(t) = 3t + t_0$, for $t < t_0$ and $x(t) = 0$ for $t \ge t_0$,

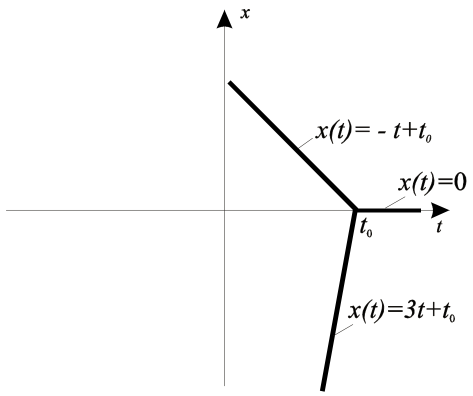

Figure3.
Solutions of the equation $\dot{x} = 1 - 2\,\mathrm{sgn}\, x$.

i.e. the classical solutions of i.v.p. (2.1) can be continuous extended from 0, obtaining the generalized solution shown in Figure 3. One of the existence assumptions is the so called growth condition

Definition 2.3. [2] The set-valued function F satisfies a **growth condition** if there exist constants $K_1, K_2 \geq 0$ with

$$\|\xi\| \leq K_1 \|x\| + K_2,$$

for all $\zeta \in F(x), x \in \mathbf{R}^n$.

Using the above results we can enounce the main result of this section

Theorem 2.4. [6] Let the i.v.p. (1.1). If g is Lipschitz continuous and satisfies the growth condition, the i.v.p. define a switch d.s.

The proof can be found in [6].

3 NUMERICAL METHODS FOR DIFFERENTIAL INCLUSIONS

In order to integrate the i.v.p. (2.4) we need to use some numerical method for differential inclusions. The simplest numerical scheme is the explicit Euler method for differential inclusions.
Let consider the general case of a nonautonomous differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad x(t_0) = x_0, \quad t \in [t_0, T], \tag{3.1}$$

A difference method consists in a replacement of the original difference inclusion (3.1) on an interval $[t_0, T]$, by a sequence of discrete inclusions on a sequence of grids of N points (see [13] and the references therein) resulting a sequence $(\eta_k)\, k = 1,2,\dots N$

$$\eta_{k+1} = \eta_k + h\xi_k, \quad \xi_k \in F(t_k, \eta_k), \quad k = 0, \cdots N-1, \quad \eta_0 = x_0. \tag{3.2}$$

The convergence results on Euler method applied to i.v.p. (3.1) can be found in [6].

Generally, the solution of the inclusion $\xi_k \in F(t_k, \eta_\kappa)$ is not unique. Then it would be selected randomly, as in the present paper, or by a suitable optimization criterion (see e.g. [15]). A Turbo Pascal code program was wrote in order to integrate and simulate the switch d.s. Three-dimensional views, phase portraits and time series can be obtained.

4 APPLICATIONS

Let consider the following discontinuous model which is a generalization of Chua's circuit [4]

$$\begin{aligned} \dot{x}_1 &= -2.57x_1 + 9x_2 + 3.86\,\mathrm{sgn}(x_1) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 \end{aligned} \quad . \tag{4.1}$$

where β is the control parameter. It is easy to see that Theorem 4.1 can be applied. Hence (4.1) defines a switch d.s. because $g = (-2.57x_1 + 9x_2, x_1 - x_2 + x_3, -\beta x_2)^T$ is linear Lipschitz continuous function and verifies the growth condition. Applying the Filippov regularization one obtain the following differential inclusion

$$\begin{aligned} \dot{x}_1 &\in -2.57x_1 + 9x_2 + 3.86\,\mathrm{Sgn}(x_1) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 \end{aligned},$$

which can be integrated using the explicit Euler method (3.2). For $\beta = 15.7$ the system behavior is chaotic (Figure 4).

The following system is a simplified model of the regulation system of a steam turbine [3]

$$\begin{aligned} \dot{x}_1 &= x_3 - x_1 - \mathrm{sgn}(x_2) \\ \dot{x}_2 &= x_1 - x_2 \\ \dot{x}_3 &= -x_2 \end{aligned} \tag{4.2}$$

Again the assumptions of Theorem 2.4 can be easily checked, the switch d.s. system (4.2) being stable.

Conclusions

In this paper we presented a class of dynamical systems, discontinuous with respect the state variable, we called switch dynamical systems. Due to the lack of solutions of the i.v.p. (1.1) we transformed the discontinuous problem into a differential inclusion which may have generalized solutions. The obtained initial value problem can be easier analyzed. The most common properties of continuous d.s. seems to be applicable to switch d.s.

The adaptation of the chaos control in switch d.s. using the OGY algorithm for continuous d.s. [14] is a subject for a future investigation.

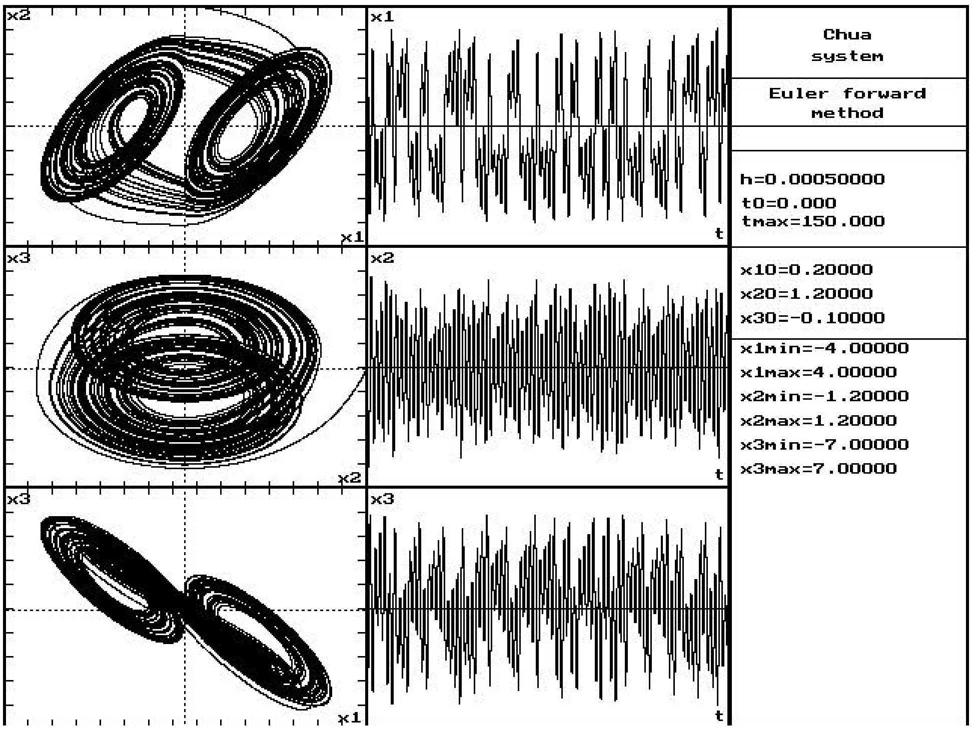

Figure 4.

A chaotic trajectory of the Chua's switch d.s.

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