AQTR 2004 (THETA 14)
2004 IEEE-TTTC - International Conference on Automation, Quality and Testing, Robotics
May 13 – 15, 2004, Cluj-Napoca, Romania

THE GENERALIZED ELASTODYNAMICS EQUATIONS IN ROBOTICS

Iuliu NEGREAN*, Darius-Gabriel ALBEȚEL**

**Technical University of Cluj-Napoca, Department of Mechanics and Computer Programming, Bd. Muncii 103-105, Cluj-Napoca, Romania, Email address: inegrean@yahoo.com*
***Technical University of Cluj-Napoca, Department of Mechanics and Computer Programming, Bd. Muncii 103-105, Cluj-Napoca, Romania, Email address: dgalbetelro@hotmail.com*

Abstract: The algorithm of the dynamic control functions *(IDM)* achieves a complex study regarding the behavior to whatever mechanical robot structure *(MRS)*, with rigid or elastic links. Beside the dynamic control functions, determined with various algorithms, the dynamic functions of the operational variables can be also analyzed. On the basis of new formulations, in this paper, the expressions for the kinetic energy, acceleration energy and generalized forces answerable to MRS with flexible links will be presented. Using on the one hand Lagrange-Euler (LE-type) equations, as well as Appell's equations, and on the other hand the Hamilton-Ostrogradski principle the generalized elastodynamics equations in Robotics will be analyzed.

Keywords: Robotics, Modeling, Applied Mechanics, Dynamics, and Elastodynamics.

1. INTRODUCTION

The dynamics robot equations can be defined, in a simplified form, as below:

$$\overline{\theta^e}(t)=f\left[\overline{Q}_m^{ee}(t)\right];\qquad \overline{Q}_m^{ee}(t)=f^{-1}\left\{\overline{\theta}^e(t);\dot{\overline{\theta}}^e(t);\ddot{\overline{\theta}}^e(t)\right\};\quad \overline{Q}_m^{ee}(t)=\left[Q_m^{iee}(t);i=1\to n\right]^T. \qquad (1)$$

Above $\left\{\overline{\theta}^e(t);\dot{\overline{\theta}}^e(t);\ddot{\overline{\theta}}^e(t)\right\}$ and $\overline{Q}_m^{ee}t)$ represent the column vectors typical of the generalized variables, and generalized driving forces respectively from every driving joint of the robot. The last is also called the dynamics control function. The mechanical robot structure *(MRS)* with *n d.o.f.,* considered as non-conservative system, has been represented in Fig.1. Unlike first equation from (1) that expresses the direct dynamics model, the second is called the inverse dynamics model *(IDM)*. The algorithm of the dynamic control functions *(IDM)* is included in *SimMEcROb Simulator* [2]. In this paper the dynamics equations for rigid structures and then the generalized elastodynamics equations for the robot with elastic links will be determined.

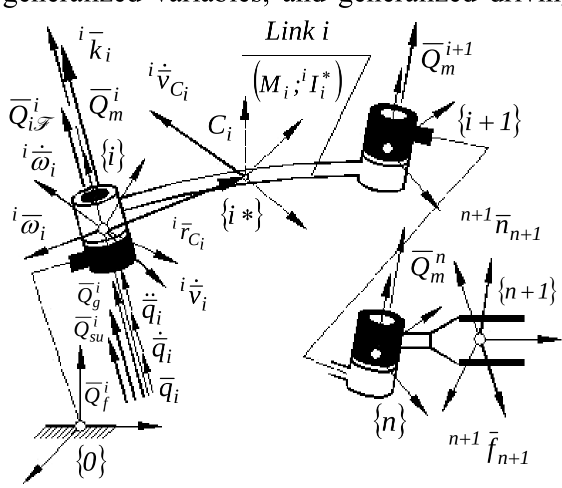

Fig. 1

2. THE MATRIX DYNAMICS EQUATIONS IN ROBOTICS

In this section, according to [2] and [3], the kinetic and acceleration energy, as well as the matrix dynamics equations for robots with rigid structure will be analyzed.

2.1 The Kinetic Energy. The Acceleration Energy

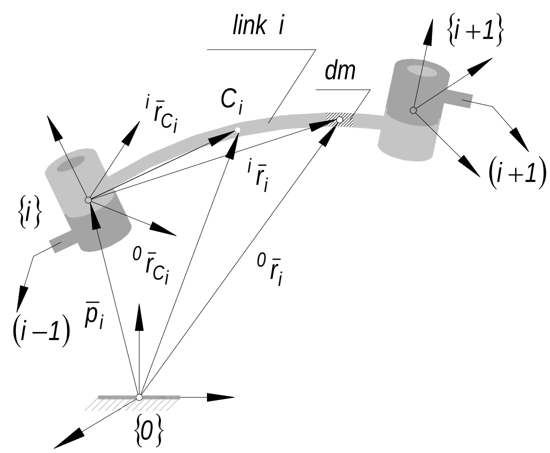

Fig. 2

In the following a kinetic link from *MRS* is taken into study, in keeping with Fig. 2. It contains infinity of elementary mass dm continually distributed in the whole volume of the kinetic link. The kinetic energy for the mechanical robot structure, having $n\ d.o.f.$, is determined with the below expression:

$$E_C\left(\bar{\theta};\dot{\bar{\theta}}\right)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{m=1}^{i}Tr\left\{A_{ij}\cdot{}^{i}I_{psi}\cdot A_{im}^{T}\right\}\dot{q}_j\cdot\dot{q}_m$$

$$E_C\left(\bar{\theta};\dot{\bar{\theta}}\right)=\left\{\begin{array}{c}\frac{1}{2}\cdot\dot{\bar{\theta}}^{T}\cdot M\left(\bar{\theta}\right)\cdot\dot{\bar{\theta}}\\ \frac{1}{2}\cdot{}^{0}\dot{\bar{X}}^{T}\cdot M_X\left(\bar{\theta}\right)\cdot{}^{0}\dot{\bar{X}}\end{array}\right\}. \qquad (2)$$

In the kinetic energy a new dynamic matrix is also defined as *the inertia matrix*:

$$M\left(\bar{\theta}\right)=\left[M_{ij}\ \begin{array}{c}i=1\rightarrow n\\ j=1\rightarrow n\end{array}\right]=\underset{(n\times n)}{Matrix}\left\{M_{ij}=M_{ji}\ \text{ where } i=1\rightarrow n \text{ and } j=1\rightarrow n\right\}; \qquad (3)$$

$$\left\{\begin{array}{c}M_{ij}=M_{ji}\\ \sum_{k=max(i;j)}^{n}Trace\left\{A_{ki}\cdot{}^{k}I_{psk}\cdot A_{kj}^{T}\right\}\end{array}\right\}=\sum_{k=max(i;j)}^{n}Tr\left\{\begin{array}{c}\left\{\left\{exp\left\{\sum_{j=0}^{i-1}A_j\cdot q_j\right\}\right\}\cdot A_i\cdot\left\{exp\left\{\sum_{l=i}^{k}A_l\cdot q_l\right\}\right\}\cdot T_{k0}^{(0)}\cdot{}^{k}I_{psk}\cdot A_{kj}^{T}\right\}\\ A_{kj}^{T}=\left\{\left\{exp\left\{\sum_{i=0}^{j-1}A_i\cdot q_i\right\}\right\}\cdot A_j\cdot\left\{exp\left\{\sum_{l=j}^{k}A_l\cdot q_l\right\}\right\}\cdot T_{k0}^{(0)}\right\}^{T}\end{array}\right\};$$

$$M_X\left(\bar{\theta}\right)={}^{0}J\left(\bar{\theta}\right)^{-T}\cdot M\left(\bar{\theta}\right)\cdot{}^{0}J\left(\bar{\theta}\right)^{-1}=\underset{(6\times 6)}{Matrix}\left\{M_{Xij}\ \text{ where } i=1\rightarrow 6 \text{ and } j=1\rightarrow 6\right\}. \qquad (4)$$

The inertia matrix (4) of the kinetic energy in the Cartesian state space is defined.

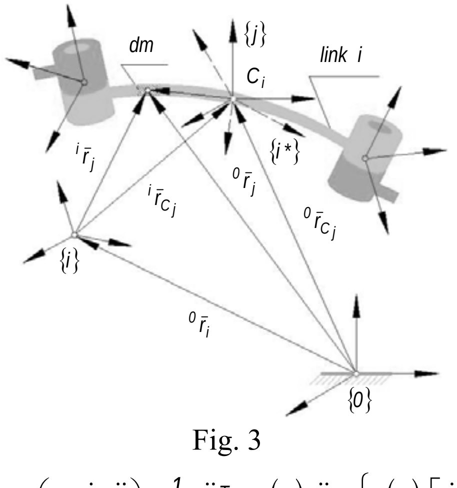

Fig. 3

It is known that, the dynamics equations to whatever mechanical robot structure *(MRS)*, can be expressed by extending the study about the *acceleration energy*. In keeping with [3], and [4], the new explicit and matrix expression of the acceleration energy to robot dynamics shows as:

$$E_A=\left\{\begin{array}{c}\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}M_{ij}\cdot\ddot{q}_i\ \ddot{q}_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}V_{ijm}\cdot\ddot{q}_i\cdot\dot{q}_j\ \dot{q}_m\\ +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}D_{ijlm}\cdot\dot{q}_i\cdot\dot{q}_j\cdot\dot{q}_l\cdot\dot{q}_m\end{array}\right\};$$

$$E_A=\frac{1}{2}\cdot\ddot{\bar{\theta}}^{T}\cdot M\left(\bar{\theta}\right)\cdot\ddot{\bar{\theta}}+V\left(\bar{\theta}\right)\cdot\ddot{\bar{\theta}}+\frac{1}{2}\cdot\dot{\bar{\theta}}^{T}\cdot D\left(\bar{\theta};\dot{\bar{\theta}}\right)\cdot\dot{\bar{\theta}};$$

$$E_A\left(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}}\right)=\frac{1}{2}\cdot\ddot{\bar{\theta}}^{T}\cdot M\left(\bar{\theta}\right)\cdot\ddot{\bar{\theta}}+\left\{B\left(\bar{\theta}\right)\cdot\left[\dot{\bar{\theta}}\dot{\bar{\theta}}\right]+C\left(\bar{\theta}\right)\cdot\left[\dot{\bar{\theta}}^{2}\right]\right\}\cdot\ddot{\bar{\theta}}+\frac{1}{2}\cdot\dot{\bar{\theta}}^{T}\cdot D\left(\bar{\theta};\dot{\bar{\theta}}\right)\cdot\dot{\bar{\theta}}; \qquad (5)$$

$$B(\overline{\theta})=\left[V_{ijm}=V_{imj}=\sum_{k=\max(i;j;m)}^{n}Tr\left[A_{ki}\cdot{}^{k}I_{psk}\cdot A_{kjm}^{T}\right]\qquad\begin{matrix}i=1\to n\\ j=1\to n-1\\ m=j+1\to n\end{matrix}\right];\tag{6}$$

$$C(\overline{\theta})=\left[V_{ijj}=\sum_{k=\max(i;j)}^{n}Tr\left[A_{ki}\cdot{}^{k}I_{psk}\cdot A_{kjj}^{T}\right]\qquad\begin{matrix}i=1\to n\\ j=1\to n\end{matrix}\right];\tag{7}$$

$$D(\overline{\theta};\dot{\overline{\theta}})=\left[\dot{\overline{\theta}}^{T}\cdot\left[D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Tr\left[A_{kij}\cdot{}^{k}I_{psk}\cdot A_{klm}^{T}\right]\quad\begin{matrix}l=1\to n\\ m=1\to n\end{matrix}\right]\cdot\dot{\overline{\theta}}\quad\begin{matrix}i=1\to n\\ j=1\to n\end{matrix}\right].\tag{8}$$

The above dynamic matrices are known as the matrix of the Coriolis terms $B(\overline{\theta})$ and centrifugal terms $C(\overline{\theta})$, while $D(\overline{\theta};\dot{\overline{\theta}})$ is a new pseudo inertia matrix. Their terms are also expressed by means of the matrix exponentials below written:

$$V_{ijm}=\sum_{k=\max(i;j;m)}^{n}Tr\left\{\begin{matrix}\left\{\exp\left\{\sum_{j=0}^{i-1}A_j\cdot q_j\right\}\right\}\cdot A_i\cdot\left\{\exp\left\{\sum_{l=i}^{k}A_l\cdot q_l\right\}\right\}\cdot T_{k0}^{(0)}\cdot{}^{k}I_{psk}\cdot A_{kjm}^{T}\\ A_{kjm}=\left\{\exp\left\{\sum_{l=0}^{m-1}A_l\cdot q_l\right\}\right\}\cdot A_m\cdot\left\{\exp\left\{\sum_{i=m}^{j-1}A_i\cdot q_i\right\}\cdot A_i\cdot\exp\left\{\sum_{p=i}^{k}A_p\cdot q_p\right\}\right\}\cdot T_{k0}^{(0)}\end{matrix}\right\};\tag{9}$$

$$D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Tr\left\{\begin{matrix}\left\{\exp\left\{\sum_{l=0}^{j-1}A_l\cdot q_l\right\}\right\}\cdot A_j\cdot\left\{\exp\left\{\sum_{m=j}^{i-1}A_m\cdot q_m\right\}\cdot A_m\cdot\exp\left\{\sum_{p=i}^{k}A_p\cdot q_p\right\}\right\}\cdot T_{k0}^{(0)}\cdot{}^{k}I_{psk}\cdot A_{klm}^{T}\\ A_{klm}=\left\{\exp\left\{\sum_{l=0}^{m-1}A_l\cdot q_l\right\}\right\}\cdot A_m\cdot\left\{\exp\left\{\sum_{i=m}^{l-1}A_i\cdot q_i\right\}\cdot A_i\cdot\exp\left\{\sum_{p=i}^{k}A_p\cdot q_p\right\}\right\}\cdot T_{k0}^{(0)}\end{matrix}\right\}.$$

The acceleration energy in the Cartesian configuration space is defined by means of the new expression:

$$E_A=\frac{1}{2}\cdot\left\{\begin{matrix}{}^{0}\ddot{\overline{X}}^{T}\cdot M_X(\overline{\theta})\cdot{}^{0}\ddot{\overline{X}}+{}^{0}\dot{\overline{X}}^{T}\cdot\left\{{}^{0}J(\overline{\theta})^{-T}\cdot{}^{0}\dot{J}(\overline{\theta})^{T}\right\}\cdot M_X(\overline{\theta})\cdot\left\{{}^{0}\dot{J}(\overline{\theta})\cdot{}^{0}J(\overline{\theta})^{-1}\right\}\cdot{}^{0}\dot{\overline{X}}+\\ +{}^{0}\ddot{\overline{X}}^{T}\cdot M_X(\overline{\theta})\cdot\left\{{}^{0}\dot{J}(\overline{\theta})\cdot{}^{0}J(\overline{\theta})^{-1}\right\}\cdot{}^{0}\dot{\overline{X}}-{}^{0}\dot{\overline{X}}^{T}\cdot\left\{{}^{0}J(\overline{\theta})^{-T}\cdot{}^{0}\dot{J}(\overline{\theta})^{T}\right\}\cdot M_X(\overline{\theta})\cdot{}^{0}\ddot{\overline{X}}+\\ +{}^{0}J(\overline{\theta})^{T}\cdot V_X(\overline{\theta};\dot{\overline{\theta}})\cdot{}^{0}\ddot{\overline{X}}-{}^{0}J(\overline{\theta})^{T}\cdot V_X(\overline{\theta};\dot{\overline{\theta}})\cdot\left\{{}^{0}\dot{J}(\overline{\theta})\cdot{}^{0}J(\overline{\theta})^{-1}\right\}\cdot{}^{0}\dot{\overline{X}}+{}^{0}\dot{\overline{X}}^{T}\cdot D_X(\overline{\theta};\dot{\overline{\theta}})\cdot{}^{0}\dot{\overline{X}}\end{matrix}\right\}.\tag{10}$$

2.2 The Dynamics Equations

On the basis of the above expressions, in keeping with [2] and [3], the column vector of the generalized variables with respect to Cartesian state space is determined:

$${}^{0}\ddot{\overline{X}}_p={}^{0}J(\overline{\theta})\cdot M(\overline{\theta})^{-1}\cdot\left\{\left\{Q_m(\overline{\theta})-B_X(\overline{\theta})\cdot\left[\dot{\overline{\theta}}\cdot\dot{\overline{\theta}}\right]-C_X(\overline{\theta})\cdot\left[\dot{\overline{\theta}}^{2}\right]\right\}-{}^{0}\overline{\mathscr{F}}_{Xg}(\overline{\theta})-{}^{0}\overline{\mathscr{F}}_{X}(\overline{\theta})\right\};\tag{11}$$

$${}^{0}\ddot{\overline{X}}_p(\tau)=\left[{}^{0}\dot{\overline{v}}_{ndp}^{T}(\tau)\quad{}^{0}\dot{\overline{\omega}}_{ndp}^{T}(\tau)\right]^{T}=\frac{\tau_p-\tau}{t_p}\cdot{}^{0}\ddot{\overline{X}}_{p-1}+\frac{\tau-\tau_{p-1}}{t_p}\cdot{}^{0}\ddot{\overline{X}}_p\text{ , and }{}^{0}\overline{X}(\tau)=\left(\overline{p}^{T}(\tau)\quad\overline{\psi}^{T}(\tau)\right)^{T};$$

$${}^{0}\overline{X}_p(\tau)=\frac{(\tau_p-\tau)^3}{6t_p}\cdot{}^{0}\ddot{\overline{X}}_{p-1}+\frac{(\tau-\tau_{p-1})^3}{6\,t_p}\cdot{}^{0}\ddot{\overline{X}}_p+\left(\frac{{}^{0}\overline{X}_p}{t_p}-\frac{t_p}{6}\cdot{}^{0}\ddot{\overline{X}}_p\right)\cdot(\tau-\tau_{p-1})+\left(\frac{{}^{0}\overline{X}_{p-1}}{t_p}-\frac{t_p}{6}\cdot{}^{0}\ddot{\overline{X}}_{p-1}\right)\cdot(\tau_p-\tau);$$

where $\tau_p \subset M_\tau$; $t_p = \tau_p - \tau_{p-1}$; ${}^0\bar{X}_p \subset DGM\ Algorithm$, and where $p = 1 \to m$ represents the number of the configurations taken into study, according to *IKM Algorithm*.

3. THE ELASTODYNAMICS EQUATIONS IN ROBOTICS

This section is devoted to define the generalized elastodynamics equations, when the robot links are dominated of flexibility properties. At first, a few kinematic transformations are described. In the aria of the small deflections, and considering the aspects from Fig.4, the time functions for the angular and linear deformations of the link (i) are written, according to [5], as:

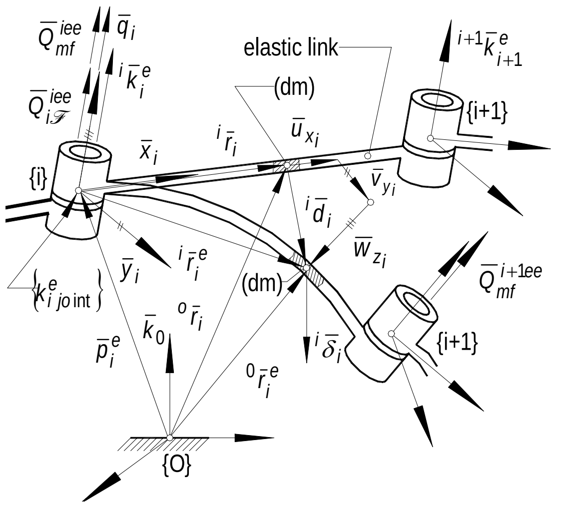

Fig. 4

$$ {}^i\bar{\delta}_i = \begin{pmatrix}\delta_{xi}\\ \delta_{yi}\\ \delta_{zi}\end{pmatrix} = \left\{\sum_{j=1}^{m_i} q_{ij}(t)\cdot {}^i\bar{\delta}_{ij}\right\} = \sum_{j=1}^{m_i} q_{ij}(t)\cdot \begin{pmatrix}\delta_{xij}\\ \delta_{yij}\\ \delta_{zij}\end{pmatrix}; $$

$$ {}^i\bar{d}_i = \begin{pmatrix}u_{xi}\\ v_{yi}\\ w_{zi}\end{pmatrix} = \left\{\sum_{j=1}^{m_i} q_{ij}(t)\cdot {}^i\bar{d}_{ij}\right\} = \sum_{j=1}^{m_i} q_{ij}(t)\cdot \begin{pmatrix}u_{ij}\\ v_{ij}\\ w_{ij}\end{pmatrix}. $$

The functions: $q_{ij}(t)$ are time amplitude of the proper modes $j = 1 \to m_i$, and they are completing the generalized variables $q_i(t)$. The position vector for an elementary mass dm is: ${}^i\bar{r}_i^e = {}^i\bar{r}_i + {}^i\bar{d}_i$; ${}^0\bar{r}_i^e = \bar{p}_i^e + R_{i0}^e \cdot \left({}^i\bar{r}_i + {}^i\bar{d}_i\right)$. The symbol (e) dignifies the elasticity of the kinetic link. After a few kinematic transformations, the new locating matrix, between adjoining elastic links, shows as:

$$ T^e_{i\,i-1} = T_{i\,i-1}\cdot \Delta T^e_{ij} = T^{ee}_{i\,i-1} + T_{i\,i-1}\left\{\sum_{j=1}^{m_i} q_{ij}\cdot \begin{bmatrix}\{\bar{\delta}_{ij}\times\} & \bar{d}_{ij}\\ 0\ 0\ 0 & 0\end{bmatrix}\right\} = T^{ee}_{i\,i-1} + T_{i\,i-1}\left\{\sum_{j=1}^{m_i} q_{ij}\cdot \Delta T_{ij}\right\}; \tag{12} $$

$$ T^e_{i0} = \prod_{j=1}^{i-1} T^e_{j\,j-1}\cdot T_{i\,i-1} = \prod_{j=1}^{i-1}\left\{T^{ee}_{j\,j-1} + T_{j\,j-1}\cdot\left\{\sum_{k=1}^{m_j} q_{jk}\cdot\begin{bmatrix}\{\bar{\delta}_{jk}\times\} & \bar{d}_{jk}\\ 0\ 0\ 0 & 0\end{bmatrix}\right\}\right\}\cdot T_{i\,i-1}; \tag{13} $$

$$ T^e_{i0} = \prod_{j=1}^{i-1} T_{j\,j-1}\cdot \Delta T^e_{jk}\cdot T_{i\,i-1} = \prod_{j=1}^{i-1}\left\{T^{ee}_{j\,j-1} + T_{j\,j-1}\cdot\left\{\sum_{k=1}^{m_j} q_{jk}\cdot \Delta T_{jk}\right\}\right\}\cdot T_{i\,i-1}. \tag{14} $$

The locating matrix $T_{i\,i-1}$ is answerable to rigid link, while ΔT^e_{ij} to the small deformations. About the above locating matrix is applied the time derivatives of first and second order as:

$$ \dot{T}^e_{i0} = \begin{bmatrix}\dot{R}^e_{i0} & \dot{\bar{p}}^e_i\\ 0\,0\,0 & 0\end{bmatrix};\ \dot{T}^e_{i0} = \sum_{k=1}^{i} T^e_{k0}\cdot U_k\cdot T^e_{ik}\cdot \dot{q}_k + \sum_{k=1}^{i-1}\sum_{l=1}^{m_k} T_{k\,k-1}\Delta T^e_{kl}\cdot T^e_{il}\cdot \dot{q}_{kl};\ \ddot{T}^e_{i0} = \begin{bmatrix}\ddot{R}^e_{i0} & \ddot{\bar{p}}^e_i\\ 0\,0\,0 & 0\end{bmatrix}. \tag{15} $$

The column vector of the generalized variables is completed with $q_{ij}(t)$ as below:

$$ \bar{\theta}^e(t) = \left[\bar{\theta}^{eT}_{ij}(t) = \left[\theta^{eT}_{ij}(t) = \begin{Bmatrix}\{q_i(t)\ if\ j = 0\}\\ \{q_{ij}(t)\ if\ j \ge 1\}\end{Bmatrix}\quad j = 0 \to m_i\right]\quad i = 1 \to n\right]^T; $$

$$ \dot{\bar{\theta}}^e(t) = \left[\dot{\bar{\theta}}^{eT}_{ij}(t) = \left[\dot{\theta}^{eT}_{ij}(t) = \left\{\{\dot{q}_i(t)\ if\ j = 0\};\{\dot{q}_{ij}(t)\ if\ j \ge 1\}\right\}\quad j = 0 \to m_i\right]\quad i = 1 \to n\right]^T; \tag{16} $$

$$\ddot{\overline{\theta}}^{e}(t)=\left[\ddot{\overline{\theta}}_{ij}^{eT}(t)=\left[\ddot{\theta}_{ij}^{eT}(t)=\left\{\{\ddot{q}_i(t)\ if\ j=0\};\ \{\ddot{q}_{ij}(t)\ if\ j\geq 1\}\right\}\quad j=0\rightarrow m_i\right]\quad i=1\rightarrow n\right]^{T}.$$

3.1 The Kinetic and Acceleration Energy in Elastodynamics

In the following, considering that *MRS* is characterized through (n) flexible links, *the kinetic energy*, which expresses the elastodynamics behavior [1], is defined as below:

$$E_C^{iee}=\frac{1}{2}\cdot\int_{link}Trace\left[\left(\dot{T}_{i0}^{e}\cdot{}^{i}\overline{r}_i+\dot{T}_{i0}^{e}\cdot{}^{i}\overline{d}_i+T_{i0}^{e}\cdot{}^{i}\dot{\overline{d}}_i\right)\cdot\left({}^{i}\overline{r}_i^{T}\cdot\dot{T}_{i0}^{eT}+{}^{i}\overline{d}_i^{T}\cdot\dot{T}_{i0}^{eT}+{}^{i}\dot{\overline{d}}_i^{T}\cdot T_{i0}^{eT}\right)\right]\cdot dm\,; \qquad (17)$$

$$E_C^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)=\left\{\begin{array}{c}\sum_{i=1}^{n}E_C^{iee}\left(\overline{\theta}_{jk}^{eT};\dot{\overline{\theta}}_{jk}^{eT};\ j=1\rightarrow i\right)\\ \frac{1}{2}\cdot\dot{\overline{\theta}}^{eT}\cdot M^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\overline{\theta}}^{e}\end{array}\right\}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}M_{ijkl}^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\overline{\theta}}_{ij}^{e}\cdot\dot{\overline{\theta}}_{kl}^{e}\,. \qquad (18)$$

➢ The elastodynamics equations can be likewise determined by means of *the Appell's equations*. In the view of this, the acceleration energy answerable to a flexible link from *MRS* is established with the new elastodynamics expression:

$$E_A^{iee}=\frac{1}{2}\cdot\int_{link}Trace\left[\left(\begin{array}{c}\ddot{T}_{i0}^{e}\cdot{}^{i}\overline{r}_i+\ddot{T}_{i0}^{e}\cdot{}^{i}\overline{d}_i+\\ +2\cdot\dot{T}_{i0}^{e}\cdot{}^{i}\dot{\overline{d}}_i+T_{i0}^{e}\cdot{}^{i}\ddot{\overline{d}}_i\end{array}\right)\cdot\left(\begin{array}{c}{}^{i}\overline{r}_i^{T}\cdot\ddot{T}_{i0}^{eT}+{}^{i}\overline{d}_i^{T}\cdot\ddot{T}_{i0}^{eT}+\\ +2\cdot{}^{i}\dot{\overline{d}}_i^{T}\cdot\dot{T}_{i0}^{eT}+{}^{i}\ddot{\overline{d}}_i^{T}\cdot T_{i0}^{eT}\end{array}\right)\right]\cdot dm\,. \qquad (19)$$

For the whole *MRS*, supposing that the (n) kinetic links are flexible *the acceleration energy* in the new matrix expression is shown in the two variants below:

$$E_A^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e};\ddot{\overline{\theta}}^{e}\right)=\frac{1}{2}\cdot\ddot{\overline{\theta}}^{eT}\cdot M^{ee}\left(\overline{\theta}^{e}\right)\cdot\ddot{\overline{\theta}}^{e}+V^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\cdot\ddot{\overline{\theta}}^{e}+\frac{1}{2}\cdot\dot{\overline{\theta}}^{eT}\cdot D^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\cdot\dot{\overline{\theta}}^{e}\,; \qquad (20)$$

$$E_A^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e};\ddot{\overline{\theta}}^{e}\right)=\left\{\begin{array}{c}\frac{1}{2}\cdot\left[\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}M_{ijkl}^{ee}\left(\overline{\theta}^{e}\right)\cdot\ddot{\overline{\theta}}_{ij}^{e}\cdot\ddot{\overline{\theta}}_{kl}^{e}+\dot{\overline{\theta}}^{eT}\cdot D^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\cdot\dot{\overline{\theta}}^{e}\right]+\\ +\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}\sum_{p=1}^{n}\sum_{r=0}^{m_p}V_{ijklpr}^{ee}\left(\overline{\theta}^{e}\right)\cdot\ddot{\overline{\theta}}_{ij}^{e}\cdot\dot{\overline{\theta}}_{kl}^{e}\cdot\dot{\overline{\theta}}_{pr}^{e}\end{array}\right\}; \qquad (21)$$

where $$\dot{\overline{\theta}}^{eT}\cdot D^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\cdot\dot{\overline{\theta}}^{e}=\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}\sum_{p=1}^{n}\sum_{r=0}^{m_p}\sum_{s=1}^{n}\sum_{u=0}^{m_s}D_{ijklprsu}^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\overline{\theta}}_{ij}^{e}\cdot\dot{\overline{\theta}}_{kl}^{e}\cdot\dot{\overline{\theta}}_{pr}^{e}\cdot\dot{\overline{\theta}}_{su}^{e}\,. \qquad (22)$$

In the above expressions (18) and (21) it remarks a few elastodynamics matrices: inertia and pseudo inertia matrices, as well as the matrices of the Coriolis and centrifugal terms.

➢ In keeping with *LE-type equations* on the one hand, and on the other hand *Appell's equations* [3] and [4], the generalized inertia forces are defined with the following:

$$Q_{i\mathscr{F}}^{kl\,ee}\left(\overline{\theta}_{kl}^{e}\right)=\left\{\frac{d}{dt}\left(\frac{\partial E_C^{ee}}{\partial\dot{\theta}_{kl}^{e}}\right)-\frac{\partial E_C^{ee}}{\partial\theta_{kl}^{e}}=\frac{\partial E_A^{ee}}{\partial\ddot{\theta}_{kl}^{e}}\right\}=\frac{d}{dt}\left\{\sum_{i=1}^{n}\sum_{j=1}^{m_i}M_{ijkl}^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\theta}_{ij}^{e}\right\}-\frac{1}{2}\cdot\dot{\overline{\theta}}^{eT}\cdot\frac{\partial}{\partial\theta_{kl}^{e}}\left\{M^{ee}\left(\overline{\theta}^{e}\right)\right\}\cdot\dot{\overline{\theta}}^{e}$$

$$\left\{\begin{array}{c}Q_{i\mathscr{F}}^{kl\,ee}=\frac{d}{dt}\left\{\sum_{i=1}^{n}\sum_{j=0}^{m_i}M_{ijkl}^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\theta}_{ij}^{e}\right\}-\frac{\partial}{\partial\theta_{kl}^{e}}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p}M_{ijpr}^{ee}\left(\overline{\theta}^{e}\right)\cdot\dot{\overline{\theta}}_{ij}^{e}\cdot\dot{\overline{\theta}}_{pr}^{e}\right\}\\ \sum_{i=1}^{n}\sum_{j=1}^{m_i}M_{ijkl}^{ee}\cdot\ddot{\theta}_{ij}^{e}+\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p}\frac{\partial}{\partial\theta_{pr}^{e}}\left\{M_{ijkl}^{ee}\right\}\cdot\dot{\theta}_{ij}^{e}\cdot\dot{\theta}_{pr}^{e}-\frac{\partial}{\partial\theta_{kl}^{e}}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p}M_{ijpr}^{ee}\cdot\dot{\theta}_{ij}^{e}\cdot\dot{\theta}_{pr}^{e}\right\}\end{array}\right\}. \qquad (23)$$

Remarks. In the above expressions, the generalized inertia forces when $l=0$ *are corresponding to generalized variable* q_k *from the driving joint, and the other to generalized variable* q_{kl} *answerable to the generalized deformations of the flexible links.*

3.2 The Elastodynamics Equations

The elastodynamics equations are devoted to establishment of the generalized driving forces for the robot structure with flexible links. In keeping with [3] and [4] all generalized elastodynamics forces are implemented with the new expressions, in which the elasticity parameters are included. The matrix equation of elastodynamics shows as:

$$Q_{i\mathscr{F}}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e};\ddot{\overline{\theta}}^{e}\right)+Q_{g}^{ee}\left(\overline{\theta}^{e}\right)+Q_{def}^{ee}\left(\overline{\theta}^{e}\right)+Q_{SU}^{ee}\left(\overline{\theta}^{e}\right)+Q_{fd}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)=Q_{mfd}\left(\overline{\theta};\dot{\overline{\theta}};\ddot{\overline{\theta}}\right). \quad (24)$$

The above equations are also obtained on the basis of *Hamilton-Ostrogradski principle*:

$$\int\left\{\delta E_{C}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)-2\cdot\frac{d}{dt}\left\{E_{C}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\right\}\cdot\delta t+\overline{Q}_{def}^{eeT}\cdot\delta\overline{\theta}_{d\delta}^{ee}+\left[\overline{Q}_{g}^{eeT}+\overline{Q}_{SU}^{eeT}-\overline{Q}_{m}^{eeT}\right]\cdot\delta\overline{\theta}^{e}\right\}\cdot dt=0\,; \quad (25)$$

$$\int\left\{\delta E_{C}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)-2\cdot\frac{d}{dt}\left\{E_{C}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e}\right)\right\}\cdot\delta t-\delta E_{P}^{ee}\left(\overline{\theta}_{d\delta}^{ee}\right)+\left[\overline{Q}_{g}^{eeT}+\overline{Q}_{SU}^{eeT}-\overline{Q}_{m}^{eeT}\right]\cdot\delta\overline{\theta}^{e}\right\}\cdot dt=0\,. \quad (26)$$

In the above equation the generalized dynamics forces $\left\{Q_{g}^{ee}\left(\overline{\theta}^{e}\right);Q_{SU}^{ee}\left(\overline{\theta}^{e}\right);Q_{def}^{ee}\left(\overline{\theta}^{e}\right)\right\}$ are implemented: They are answerable to the gravity forces, manipulating load; deformations of the flexible structure, and $Q_{m}^{ee}\left(\overline{\theta}^{e};\dot{\overline{\theta}}^{e};\ddot{\overline{\theta}}^{e}\right)$ is the column vector of the generalized driving forces from every driving joint of the elastic robot structure.

4. CONCLUSIONS

Within of this paper, the generalized elastodynamics equations have been analyzed for the robot structure with flexible links. At first the kinetic and acceleration energy for rigid structures have been defined by means of the matrix exponentials. On the basis of a few formulations, the defining expressions to generalized elastodynamics forces have been determined. In the view of this, the matrix expression of the kinetic and then acceleration energy has been presented. In the study of the mechanical robot structures with elastic driving joints and flexible links, the influences of the small linear and angular deformations about the generalized inertia and active forces have been dignified. The algorithm of the generalized elastodynamics forces, described in this paper, will be included into *SimMEcROb Simulator* devoted to complex study concerning geometry, kinematics, dynamics, as well as accuracy for the robots with rigid and flexible links.

REFERENCES

[1] Book, W. J., (1984), Recursive Lagrangian Dynamics of Flexible Manipulator Arms, *In: International Journal of Robotics Research,* Vol. 3, pp.87-101.
[2] Negrean, I., (1999), Kinematics and Dynamics of Robots—Modelling • Experiment • Accuracy, *Editura Didactica si Pedagogica R.A.,* Bucharest, Romania.
[3] Negrean, I. and Negrean, D. C., (2002), The Acceleration Energy to Robot Dynamics, *In: the Proceedings of A&QT-R (THETA 13) International Conference on Automation, Quality and Testing, Robotics,* Cluj-Napoca, Romania, Tome II, pp.53-58.
[4] Negrean, I., Negrean, D. C. and Albeţel, D. G., (2003), An Approach of the Generalized Forces in the Robot Control, *Proceedings, CSCS-14, 14th International Conference on control Systems and Computer Science*, Polytechnic University of Bucharest, ISBN 973-8449-17-0, pp. 131-136.
[5] Yuan, B.–S., Book, W. J. and Huggins, J. D., (1993), Dynamics of Flexible Manipulator Arms: Alternative derivation, Verification, and Characteristics for Control, *ASME Journal of Dynamics Systems, Measurements and Control*, Vol. 115, pp.394-404.