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VARIANT OF START THROUGH TAYLOR SERIES FOR NUMERICAL INTEGRATION THROUGH L.L.I. OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In present work, are exposed the main steps, which are necessary to assure the start of calculus, for numerical integration of partial differentials equations (pde), through Local Iterative Linearization method (L.I.L.). Because the method is of multi-step type, limited at three or five regressive sequences, is not possible to make a self-start of numerical integration. The start for partial differential equation numerical integration through Runge – Kutta methods, was proved to be too laborious or not practice, in papers [….] was used, with good results, the start through Taylor series. As follow of this conclusions, this work present main steps of start through Taylor series, exemplified for a two order partial differentials equation, with two independent variables.

Keywords
partial differential equations, initial conditions, boundary conditions, state variables, complementary variables, numerical integration

1. INTRODUCTION

It is considered pdeII/2, very useful in technical and scientific applications:

$$P_{00}\cdot y + P_{10}\cdot\frac{\partial y}{\partial a} + P_{01}\cdot\frac{\partial y}{\partial b} + P_{20}\cdot\frac{\partial^2 y}{\partial a^2} + P_{11}\cdot\frac{\partial^2 y}{\partial a\partial b} + P_{02}\cdot\frac{\partial^2 y}{\partial b^2} = \varphi(a,b),(1)$$

where the solution $y = y(a,b)$ and right member $\varphi(a,b)$ are continue functions, and the coefficients $P_{\dots}$ are considered constant. Numerical integration are make by (a) variable, which can be, as example time, and (b) variable can be a spatial dimension (length, width, height, …). Such follow, $y_{IC} = y(a_0,b)$ belong to known initial conditions, and $y_{FC1} = y(a,b_0)$, $y_{FC2} = y(a_f,b)$ and $y_{FC3} = y(a,b_f)$ respectively belong to boundary conditions, respectively: input, final and output, such that are exemplified in Fig. 1.

Using the notation:

$$x_{pq} = \frac{\partial^{p+q} y}{\partial^p a \partial^q b}, \ (p = 0, 1, 2; \ q = 0, 1, 2), \qquad (2)$$

result that (1) can be rewrite in form:

$$P_{00}x_{00} + P_{10}x_{10} + P_{01}x_{01} + P_{20}x_{20} + P_{11}x_{11} + P_{02}x_{02} = \varphi(a,b) \qquad (3)$$

in comparison with which, according to LIL method, are defined:

$$\mathbf{x} = \boxed{\begin{array}{c} x_{00} \\ \hline x_{10} \end{array}} \qquad (4)$$

and

$$\tilde{\mathbf{x}} = \boxed{\begin{array}{c} x_{01} \\ \hline x_{11} \\ \hline x_{02} \end{array}} \qquad (5)$$

the state vector $\mathbf{x}(2\times1)$ and the complementary state vector $\tilde{\mathbf{x}}\,(3\times1)$.

Because this method of integration is a multi-step type, to assure start of calculations, are necessary three (rarely five) regressive sequences, noted by the indices (k-j), for j = 1,2,3 (rarely j = 1,2,3,4,5). The (k) index it correspond to current sequence, associate to pivot variable (a_k), in proximity which ($a_k \pm \delta a$) are operated numerical integration, with integration step $\Delta a = 2\cdot\delta a$.

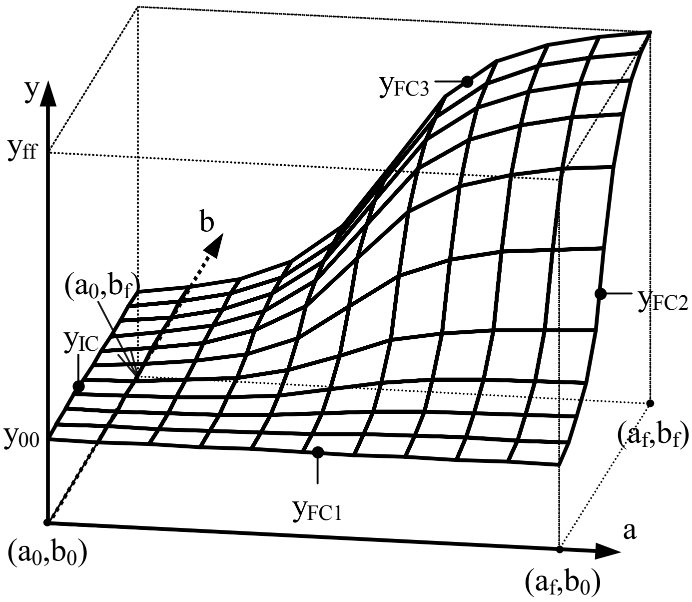

Fig. 1

2. USING OF TAYLOR SERIES FOR START OF INTEGRATION

Because the initial conditions (IC) are defined by closed segment of abscise, delimited by points (a_0,b_0) and (a_0,b_f), exemplified in Fig. 1, then result:

$$\mathbf{x}_{IC} = \boxed{\begin{array}{c} x_{00IC} = x_{00}(0,b) \\ \hline x_{10IC} = x_{10}(0,b) \end{array}} \ \text{and} \ \tilde{\mathbf{x}}_{IC} = \boxed{\begin{array}{c} x_{01IC} = x_{01}(0,b) \\ \hline x_{11IC} = x_{11}(0,b) \\ \hline x_{02IC} = x_{02}(0,b) \end{array}}, \qquad (6 \text{ and } 7)$$

where, for simplifying, are considered $a_0 = 0$.

It follows that for usual case of three regressive sequences, ca is written:

$$\mathbf{x}_{IC,k-1} = \mathbf{x}_{IC}; \ \tilde{\mathbf{x}}_{IC,k-1} = \tilde{\mathbf{x}}_{IC}; \qquad (8 \text{ and } 9)$$

$$\mathbf{x}_{IC,k-2} \cong \mathbf{x}_{IC} + \sum_{p=1}^{4} \frac{(-\Delta a)^p}{p!} \cdot \overset{p\cdot a}{\mathbf{x}_{IC}}; \ \tilde{\mathbf{x}}_{IC,k-2} = \tilde{\mathbf{x}}_{IC} + \sum_{p=1}^{4} \frac{(-\Delta a)^p}{p!} \cdot \overset{p\cdot a}{\tilde{\mathbf{x}}_{IC}}; \qquad (10 \text{ and } 11)$$

$$\mathbf{x}_{IC,k-3} \cong \mathbf{x}_{IC} + \sum_{p=1}^{4} \frac{(-2\Delta a)^p}{p!} \cdot \overset{p\cdot a}{\mathbf{x}_{IC}}; \ \tilde{\mathbf{x}}_{IC,k-3} = \tilde{\mathbf{x}}_{IC} + \sum_{p=1}^{4} \frac{(-\Delta a)^p}{p!} \cdot \overset{p\cdot a}{\tilde{\mathbf{x}}_{IC}}; \qquad (12 \text{ and } 13)$$

Taylor series successive refining from above was limited to differential of order p = 4 inclusively, assuring that a enough precision for stability of regressive sequences (k-2) and (k-3).

3. STAGES OF CALCULUS

Stage 1.
From known initial conditions (IC), defined in (6), are calculated, through analytical derivation by (b) the elements of state complementary vector $\tilde{\mathbf{x}}_{\mathbf{IC}}$ defined in (7).
Stage 2.
Are make analytical derivation successively, for p = 1, 2, 3, 4 times by (a) the function $\varphi(a,b)$ from (1), then result $\overset{pa}{\varphi}(a,b)$, for $a = a_0 = 0$.
Are make analytical derivation successively, for q = 1, 2, 3, 4 times by (b) the function $\varphi(a,b)$ from (1), then result $\overset{qb}{\varphi}(a,b)$, for $a = a_0 = 0$.
Formally identical are calculated $\overset{a,qb}{\varphi}(a,b)$ for q = 1, 2, 3 and, after that, $\overset{2a,qb}{\varphi}(a,b)$ for q = 1, 2, and $\overset{3a,1b}{\varphi}(a,b)$, all for $a = a_0 = 0$.
Stage 3.
Are successively analytical differential by (b), the vectors $\mathbf{x}_{\mathbf{IC}}$ and $\tilde{\mathbf{x}}_{\mathbf{IC}}$ from (6) and (7) respectively, then result:

$$\overset{q\cdot b}{\mathbf{x}}_{\mathbf{IC}} = \left.\begin{array}{|c|}\hline x_{0,q}\\ \hline x_{1,q}\\ \hline\end{array}\right|_{a=a_0=0}, \qquad \text{for } q = 1, 2, \dots 6. \tag{14}$$

The results of analytical differentials from above, is used in following stages, then the operations of derivations by (a) and (b) will become symbolic and can be used in logical schemes of programming, with same precision for IC, respectively $a = a_0 = 0$.

Stage 4. It It is calculated:

$$\overset{a}{\mathbf{x}}_{\mathbf{IC}} = \left.\begin{array}{|c|}\hline x_{10}\\ \hline x_{20} = \dfrac{1}{P_{20}}\cdot\left[\varphi(a,b) - (P_{00}x_{00} + P_{10}x_{10} + P_{01}x_{01} + P_{11}x_{11} + P_{02}x_{02})\right]\\ \hline\end{array}\right|_{a=a_0=0}, \tag{15}$$

$$\overset{a,q\cdot b}{\mathbf{x}}_{\mathbf{IC}} = \left.\begin{array}{|c|}\hline x_{1,q}\\ \hline x_{2,q} = \dfrac{1}{P_{20}}\cdot\left[\overset{q\cdot b}{\varphi}(a,b) - (P_{00}x_{0,q} + P_{10}x_{1,q} + P_{01}x_{0,1+q} + P_{11}x_{1,1+q} + P_{02}x_{0,2+q})\right]\\ \hline\end{array}\right|_{a=a_0=0}, \tag{16}$$

for q = 1, 2, 3, 4.
Stage 5. It It is calculated:

$$\overset{2a}{\mathbf{x}}_{\mathbf{IC}} = \left.\begin{array}{|c|}\hline x_{20}\\ \hline x_{30} = \dfrac{1}{P_{20}}\cdot\left[\overset{a}{\varphi}(a,b) - (P_{00}x_{10} + P_{10}x_{20} + P_{01}x_{11} + P_{11}x_{21} + P_{02}x_{12})\right]\\ \hline\end{array}\right|_{a=a_0=0}, \tag{17}$$

$$\overset{2a,q\cdot b}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{2,q} \\ \hline x_{3,q} = \frac{1}{P_{20}} \cdot \left[\overset{a,q\cdot b}{\varphi}(a,b) - (P_{00}x_{1,q} + P_{10}x_{2,q} + P_{01}x_{1,1+q} + P_{11}x_{2,1+q} + P_{02}x_{1,2+q}) \right] \end{array}}\right|_{a=a_0=0}, \quad (18)$$

for q = 1, 2, 3.

Stage 6. It It is calculated:

$$\overset{3a}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{30} \\ \hline x_{40} = \frac{1}{P_{20}} \cdot \left[\overset{2a}{\varphi}(a,b) - (P_{00}x_{20} + P_{10}x_{30} + P_{01}x_{21} + P_{11}x_{31} + P_{02}x_{22}) \right] \end{array}}\right|_{a=a_0=0}, \quad (19)$$

$$\overset{3a,q\cdot b}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{3,q} \\ \hline x_{4,q} = \frac{1}{P_{20}} \cdot \left[\overset{2a,q\cdot b}{\varphi}(a,b) - (P_{00}x_{2,q} + P_{10}x_{3,q} + P_{01}x_{2,1+q} + P_{11}x_{3,1+q} + P_{02}x_{2,2+q}) \right] \end{array}}\right|_{a=a_0=0}, \quad (20)$$

for q = 1, 2.

Stage 7. It is calculated:

$$\overset{4a}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{40} \\ \hline x_{50} = \frac{1}{P_{20}} \cdot \left[\overset{3a}{\varphi}(a,b) - (P_{00}x_{30} + P_{10}x_{40} + P_{01}x_{31} + P_{11}x_{41} + P_{02}x_{32}) \right] \end{array}}\right|_{a=a_0=0}, \quad (21)$$

$$\overset{4a,q\cdot b}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{4,q} \\ \hline x_{5,q} = \frac{1}{P_{20}} \cdot \left[\overset{3a,q\cdot b}{\varphi}(a,b) - (P_{00}x_{3,q} + P_{10}x_{4,q} + P_{01}x_{3,1+q} + P_{11}x_{4,1+q} + P_{02}x_{3,2+q}) \right] \end{array}}\right|_{a=a_0=0}, \quad (22)$$

for q = 1.

Stage 8. It is calculated:

$$\overset{5a}{\mathbf{x}_{IC}} = \left.\boxed{\begin{array}{c} x_{50} \\ \hline x_{60} = \frac{1}{P_{20}} \cdot \left[\overset{4a}{\varphi}(a,b) - (P_{00}x_{40} + P_{10}x_{50} + P_{01}x_{41} + P_{11}x_{51} + P_{02}x_{42}) \right] \end{array}}\right|_{a=a_0=0}, \quad (23)$$

It can be observed that after repeated analytical derivations from stages 1, 2 and 3, in following stages of symbolic derivations, 4, 5, 6, 7 and 8 respectively, every differential ($x_{p,q}$) result from anterior stages of calculus.

Sow, these results are preloaded automatically in programming lines which it follow, facilitating speed of this calculus.

With results obtained, can be calculated the regressive sequences (8-13), that is necessary to assure start of integration, through L.I.L of pde (1).

All stages from above, included in (4-23), can be presented, much condensed and resumed, in follow succession, all for IC ($a = a_0 = 0$):

a) $\overset{pa}{\varphi}(a,b)$, (p = 1, 2, 3, 4);

b) $\overset{qb}{\varphi}(a,b)$, (q = 1, 2, 3, 4);

c) $\overset{a,qb}{\varphi}(a,b)$, (q = 1, 2, 3); $\overset{2a,qb}{\varphi}(a,b)$, (q = 1, 2); $\overset{3a,1b}{\varphi}(a,b)$;

All three grouped expression from above are based at analytical derivations, after the variables (a) or (b).

Next derivations are symbolical, operated at state vector $\mathbf{x_{IC}}$ known from (6), respectively:

d) $\overset{q\cdot b}{\mathbf{x_{IC}}}$, (q = 1, 2, … 6);

e) $\overset{a}{\mathbf{x_{IC}}}$ and $\overset{a,q\cdot b}{\mathbf{x_{IC}}}$, (q = 1, 2, 3, 4);

f) $\overset{2a}{\mathbf{x_{IC}}}$ and $\overset{2a,q\cdot b}{\mathbf{x_{IC}}}$, (q = 1, 2, 3);

g) $\overset{3a}{\mathbf{x_{IC}}}$ and $\overset{3a,q\cdot b}{\mathbf{x_{IC}}}$, (q = 1, 2);

h) $\overset{4a}{\mathbf{x_{IC}}}$ and $\overset{4a,q\cdot b}{\mathbf{x_{IC}}}$, (q = 1);

i) $\overset{5a}{\mathbf{x_{IC}}}$.

As a remark, only the points a-d) are dependent by expressions of right member φ(a,b) and vector $\mathbf{x_{IC}}$ belonging to pde (1). The points d-i) is program lines which remain unchanged, facilitating simplicity of start algorithm through Taylor series.

4. EXAMPLE OF START FOR PDE, THROUGH TAYLOR SERIES

Are considered pde (1), which in hypothesis that admit analytical solution:

$$x_{00AN} = y(a,b) = J_0 + J_1\cdot \varepsilon^{\alpha a+\beta b}, \tag{24}$$

it result in form:

$$P_{00}\cdot y + P_{10}\cdot\frac{\partial y}{\partial a} + P_{01}\cdot\frac{\partial y}{\partial b} + P_{20}\cdot\frac{\partial^2 y}{\partial a^2} + P_{11}\cdot\frac{\partial^2 y}{\partial a\partial b} + P_{02}\cdot\frac{\partial^2 y}{\partial b^2} =$$

$$P_{00}\cdot J_0 + (P_{00} + P_{10}\cdot\alpha + P_{01}\cdot\beta + P_{20}\cdot\alpha^2 + P_{11}\cdot\alpha\beta + P_{02}\cdot\beta^2)\cdot J_1\cdot \varepsilon^{\alpha a+\beta b} = \varphi(a,b) \tag{25}$$

If: $P_{00} = 1$, $P_{10} = 2$, $P_{01} = 3$, $P_{20} = 4$, $P_{11} = 5$, $P_{02} = 6$, $J_0 = 1$, $J_1 = 1$, $\alpha = -10^{-1}$, $\beta = -10^{-1}$ then

$$\varphi(a,b) = J_{0\varphi} + J_{1\varphi}\cdot \varepsilon^{\alpha a+\beta b}, \tag{26}$$

where $J_{0\varphi} = 1$ and $J_{1\varphi} = 0.65$.

Is followed the point's a-c), all for IC ($a = a_0 = 0$):

a) $\overset{pa}{\varphi}(a,b) = \alpha^p\cdot J_{1\varphi}\cdot \varepsilon^{\alpha a+\beta b}$, (p = 1, 2, 3, 4);

b) $\overset{qb}{\varphi}(a,b) = \beta^q\cdot J_{1\varphi}\cdot \varepsilon^{\alpha a+\beta b}$, (q = 1, 2, 3, 4);

c) $\overset{pa,qb}{\varphi}(a,b) = \alpha^p\cdot\beta^q\cdot J_{1\varphi}\cdot \varepsilon^{\alpha a+\beta b}$, (p = 1, q = 1, 2, 3), (p = 2, q = 1, 2), (p = 3, q = 1).

The initial conditions IC ($a = a_0 = 0$) for state vector, is in form:

$$\mathbf{x_{IC}} = \boxed{\begin{array}{l} x_{00IC} = J_0 + J_1\cdot \varepsilon^{\alpha a+\beta b} \\ \hline x_{10IC} = \alpha\cdot J_1\cdot \varepsilon^{\alpha a+\beta b} \end{array}}\ . \tag{27}$$

As continuing, follow walking (through programming) of symbolic derivation stages 4-8, using vectored relations (15-23). With results thus obtained It is calculated the regressive sequences (8-13), necessary to assure start of calculus through L.I.L. method.

In basis of presented algorithm with 8 stages, together with groups a-i), was drawled out the program (START0001(2)), from which was extracted some results, for example (24-27), exposed in Table 1.

Table 1.

Δa	a_0	$b_k b_0$	P_{00}	P_{10}	P_{01}	P_{20}	P_{11}	P_{02}	J_0	J_1	α	B	x_{ka-1}	x_{ka-2}	x_{ka-3}
0.1	0	0	1	2	3	4	5	6	1	1	-0.1	-0.2	2	2.01005	2.020202
0.05	0	10	6	5	4	3	2	1	10	-10	-0.01	-0.5	9.93261	9.932587	1.932553
0.03	10	0	3	5	1	8	4	2	-5	10	-0.2	-0.01	-3.646647	-3.638503	-3.630309
0.5	-10	10	10	1	6	3	4	8	10	-5	-0.05	-0.1	6.967347	6.890575	6.811859

For all combinations, desired artificially, presented in Table 1, all obtained values (x_{ka-1}, x_{ka-2} and x_{ka-3}) was compared with analytical solution (24), then result practically overlapping of resulted values. Relative errors in percents (erp), defined through:

$$\text{erp} = \frac{x_{ka-j} - x_{ka-jAN}}{x_{ka-j}} \cdot 100\,, \tag{28}$$

(for j = 1, 2, 3) do not obey, in module 10^{-6} %.

5. CONCLUSIONS

Variant of start through Taylor series, for numerical integration through L.I.L. of partial differential equations, exposed in this work, is simple, performing and with a high rank of generalizing, for a large diversity of this types of equations.

Program makes (START0001(2)) it considered $a_0 \neq 0$ case inclusively, $a_0 = \pm 10$ respectively, and ($0.03 \leq \Delta a \leq 0.5$).

The first two lines from Table 1, it correspond to a hyperbolic pde ($P_{11}^2 - P_{20} \cdot P_{02} = 0$), and elliptic respectively ($P_{11}^2 - P_{20}P_{02} < 0$). Then result that this program can be easy adapted for numerous pde, with different initial conditions.

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