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SOME ASPECTS CONCERNING THE OPTIMAL CONTROL OF HYBRID SYSTEMS

Bogdan Droaşcă[1], **Virginia Ecaterina Oltean**[1] **and Horia Andrei**[2]

"Politehnica" University of Bucharest, Faculty of Control and Computers[1], *Faculty of Electrotechnics*[2], *313 Spl.Independenţei, sect. 6, 77206 Bucharest, ROMANIA*
bogdan_droasca@yahoo.com, oltean@aii.pub.ro

ABSTRACT. Hybrid systems have received a lot of attention in the past decade and a number of different models have been proposed in order to establish mathematical framework that is able to handle both continuous and discrete aspects. This paper is focused on two classes of model: hybrid automata and hybrid control systems with a separable cost structure, allowing the decomposition into a lower level part, with time-driven dynamics interacting with a higher-level component, with event-driven dynamics. A time-optimal control problem can be formulated within a hybrid automata model. The second class of hybrid models ensures the performance optimization of both hierarchical components, thus resulting an optimal switching control policy. Specific aspects of both approaches are comparatively emphasized.

KEYWORDS: discrete event system (DES), hybrid systems, hybrid automaton, hybrid controller, optimal control.

1. INTRODUCTION

The term "hybrid" is used to characterize systems that combine time-drive and event-driven dynamics. The former are represented by differential (or difference) equations, while the latter may be described through various frameworks used for discrete event systems (DES) [2], like an automaton or an input-output transition system with a finite number of states [1], [4]. Broadly speaking, two categories of modeling frameworks have been proposed to study hybrid systems: those that extend event-driven models to include time-driven dynamics and those that extend the traditional time-driven models to include event-driven dynamics [5]. Thus, hybrid systems are systems that involve both continuous and discrete variables and their evolution is given by state equations that generally depend on all the variables. Hybrid systems have received a lot of attention in the past decade and a number of different models have been proposed in order to establish mathematical framework that is able to handle both continuous and discrete aspects.

This contribution is focused on two classes of models: hybrid automata, [1], [4] and hybrid control systems with a separable cost structure, allowing the decomposition into a lower level part, with time-driven dynamics interacting with a higher-level component, with event-driven dynamics [3], [5]. A hybrid automaton represents essentially a set of continuous dynamic systems, together with a switching policy among these continuous dynamics. A time-optimal control problem can be formulated within a hybrid automata model. The second class of hybrid models ensures the performance optimization of both hierarchical components, thus resulting an optimal switching control policy; this framework was successfully applied to methalurgical manufacturing systems, among others [3].

Section 2 presents a brief overview of the hybrid automata framework [1], [4] and introduces the time optimal control problem. Section 3 discusses the optimal control problem for a hybrid system with *separabable cost structure* and briefly presents the related hierarchical decomposition, thus resulting a hierarchical two-levels hybrid controller [3], [5]. The plant coupled to this controller thus becomes a classic continuous system with an optimal switching control policy. At this point, a possible conversion to hybrid automata models may lead to an interesting comparison between the two optimal control problems.

2. HYBRID AUTOMATA – A BRIEF OVERVIEW AND RELATED PROBLEMS

2.1 Review on hybrid automata

The framework used here is based on the hybrid automata first introduced in [1]. A hybrid system is modeled as a finite automaton that is equipped with a set of variables. In each location of the automaton, the values of the variables change continuously with time, according to a specific evolution law. Each transition of the automaton is guarded by an event and its execution modifies the values of the variables, according to the associated assignment. Each location is also labeled with an invariant condition that must hold as long as the system resides at a location.

A *hybrid automaton H* is a seven-tuple $H = (Var, Loc, Act, Inv, A, Ev, Ass)$ where: (1) *Var* is the finite number of real-valued variables x^1, x^2, …, x^n. The continuous state of *H* is $x = [x^1 \; x^2 \; \dots \; x^n]^T \in \mathbf{R}^n$ and it is also the variable state of *H*. (2) *Loc* is a finite set of vertices called *locations*; (3) *Act* is a function that assigns to each location $l \in Loc$ a function act_l describing the evolution of variables according with time. In general, act_l can be defined as an evolution law: $act_l : \dot{x} = f(x, u)$. (4) *A* is a finite set of arcs called *transitions*. Each transition $a = (l, l')$ identifies a source location $l \in Loc$ and a target location $l' \in Loc$. (5) *Ev* is a function that assigns to each transition $a = (l, l')$ a predicate Ev_a, called event. The execution of the transition $a = (l, l')$ is conditioned by the occurrence of the event Ev_a. (6) *Ass* is a function that assigns to each transition $a = (l, l')$ a relation Ass_a called *assignment*. It is used for updating the variable state and generally defined as: Ass_a: x:=$g(x)$. □

At any instant, the *state* of a hybrid system is given by a location $l \in Loc$ and a continuous (or variable) state $x \in \mathbf{R}^n$, so the system state is characterized by the pair $s = (l, x)$. It can change in two ways: (*i*) by a *time delay* measured by a *clock*, that changes only the variable state according to the evolution law associated with current location; such a delay can take place as long as the variable state satisfies the location invariant; (*ii*) by the *instantaneous execution of a transition* that changes both the location and current variable state according to the transition assignment. The execution of the transition is conditioned by the occurrence of the associated event.

If in a location l $act_l : \dot{x} = 1$ and for all transitions having the source in *l*, $a = (l, l')$ with $l' \in Loc$, the assignment is defined by: Ass_a: $x := g(x) \in \{0, x\}$, then *x* is a *clock*. Thus: (1) the value of the clock increases uniformly with time, and (2) a discrete transition either resets a clock to 0 or leaves it unchanged. If in the above definition $act_l : \dot{x} = k$, with $k \in \mathbf{Z}$ an integer nonzero constant, and for all transitions having the source in *l*, $a = (l, l')$ with $l' \in Loc$, the assignment is defined by: Ass_a: $x := g(x) \in \{0, x\}$, then *x* is a *skewed clock*.

2.2 *Specific problems*

We shall first briefly present an example, introduced in [4].

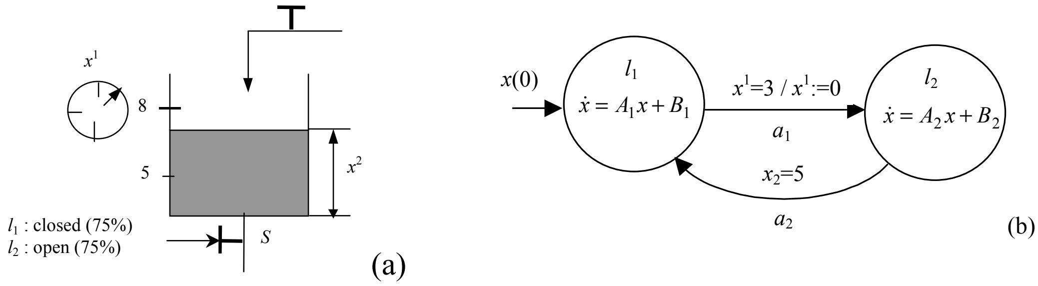

Figure 1. Hybrid system representing the controlled level dynamics in a water tank (a); the corresponding hybrid automaton (b).

Consider the hybrid automaton in fig. 1(b). The level x^2 in the water tank in Figure 1(a) is controlled by an automaton that measures it and opens or closes the valve S. The state variable x^1 represents the clock of the controller ($\dot{x}^1 = 0$). When the controller closes the valve (location l_1 in the hybrid automaton), then the level rises with the speed $\dot{x}^2 = -0.5x^2 + 5.15$. When the valve is open (location l2 in Figure 1(b)), then the level goes down with the speed $\dot{x}^2 = -0.4x^2 + 1$. In each location of the hybrid automaton, the evolution laws are represented by linear state equations, with the matrices:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 5.15 \end{bmatrix};\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -0.4 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (1)$$

The hybrid system has the following evolution. Suppose that $x(0) = [1.33\ 5]^T$ and the valve S is closed (location l_1). When the clock x^1 reaches the value 3, then the controller opens the valve S (the transition a_1 is fired) and resets the clock x^1 to zero. In the location l_2, when the water level x^1 reaches the value 5, then the controller closes the valve S (transition a_2 is fired) and the hybrid automaton returns to location l_1. □

The hybrid automata approach is basically dedicated to systems analysis, and a relevant problem is the *analysis of the reachability* of a given global state. Let s and s' be two global states of *H*. The state s' is reachable from the state *s*, $s \mapsto s'$, if there is a *run* of *H* (i.e., informally, an *admissible* sequence of global states of *H*) that starts in *s* and ends in s'.

Additionally, the following *optimal control problem* may be formulated: given a linear hybrid automaton with linear evolutions in each location,

$$\dot{x} = A_j x + B_j u ,\ \forall l_j \in Loc , \tag{2}$$

together with an initial state $s^0 = (l_0, x^0)$ and a desired (reachable) final state $s^F = (l_F, x^F)$, find the control $u(\cdot)$ that minimizes the functional

$$J = \sum_{\forall i} \delta_i , \tag{3}$$

where $\delta_i > 0$ represents the time interval for which the hybrid systems resides within location $l_i \in Loc$, in its discrete evolution.

3. OPTIMAL CONTROL OF HYBRID SYSTYEMS WITH HIERARCHICAL DECOMPOSITION

3.1 The problem formulation

In the hybrid systems framework introduced in [3] and [5], the state of the system consists of temporal and physical components. The temporal components keep track of the time information for systems events that may cause switches in the operating mode of the system. Let $i = 1,2,\ldots$ index these events. The *i*th *physical state* of the system is denoted by $z_i(t)$, with the dynamics

$$\dot{x}_i = g_i(x_i, u_i, t),\ x_i(e_{i-1}) = x_i^0 , \tag{4}$$

where $u_i(t)$ is the control applied over an interval $[e_{i-1}, e_i)$ defined by two event occurrences at times e_{i-1} and e_i. In the case of a single event-driven process in the system, the *event-driven dynamics* characterizing the temporal state e_i are given by

$$e_i = e_{i-1} + \gamma_i(x_i, u_i) , \tag{5}$$

for $i = 1,2,\ldots$, where $\gamma_i(x_i, u_i)$ represents the amount of time between switches, which generally depends on the physical state and control.

The *optimal control problem* considered is

$$\min_{\mathbf{u}} J = \sum_{i=1}^{N} [\phi_i(x_i, u_i, e_i, e_{i-1}) + \psi_i(e_i)] . \tag{6}$$

Here, $\phi_i(x_i, u_i, e_i, e_{i-1})$ is the cost of operating the system with control $u_i(t)$ resulting with the physical state $x_i(t)$ over interval $[e_{i-1}, e_i)$, and $\psi_i(e_i)$ is the cost associated with the occurrence at time e_i of the *i*th event. Assumming that $\phi_i(x_i, u_i, e_i, e_{i-1}) = \phi_i(x_i, u_i, \delta_i)$, with $\delta_i = e_i - e_{i-1} \geq 0$, $i = 1,2,\ldots$, the optimal control problem can be written

$$\min_{\mathbf{u}} J = \sum_{i=1}^{N} [\phi_i(x_i, u_i, \delta_i) + \psi_i(e_i)] . \tag{7}$$

subject to (4) and (5), where $\mathbf{u} = \{u_1, \ldots, u_N\}$ and N is the number of time-intervals involved in the operation of the system.

3.2 Hierarchical decomposition

Since $u_i(t)$, $t \in [e_{i-1}, e_i)$ is a function of $x_i(t)$ and δ_i, (7) can be rewritten as

$$\min_{\mathbf{x}^0, \mathbf{x}^F, \boldsymbol{\delta}} \sum_{i=1}^{N} [\min_{u_i(x_i^0, x_i^F, s_i)} \phi_i(x_i, u_i, \delta_i) + \psi_i(e_i)]. \tag{8}$$

Further, it can be imposed a *decomposition* into a *collection of inner minimization problems*

$$\theta_i(x_i^0, x_i^F, \delta_i) \equiv \min_{u_i} \phi_i(x_i, u_i, \delta_i), \tag{9}$$

subject to (4) for all $i = 1, 2, \ldots$, with the (time-varying) solutions, respectively,

$$u_i^*(x_i^0, x_i^F, \delta_i) \equiv \arg\min_{u_i} \phi_i(x_i, u_i, \delta_i), \tag{10}$$

and the *outer minimization problem*

$$\min_{\mathbf{x}^0, \mathbf{x}^F, \boldsymbol{\delta}} \sum_{i=1}^{N} [\theta_i(x_i^0, x_i^F, \delta_i) + \psi_i(e_i)], \tag{11}$$

subject to (5). Once the optimal time events determined from (11), (10) is used to determine the N optimal controls in the operation of the system. The hybrid controller for coordinating the two problems has the structure depicted in Figure 2.

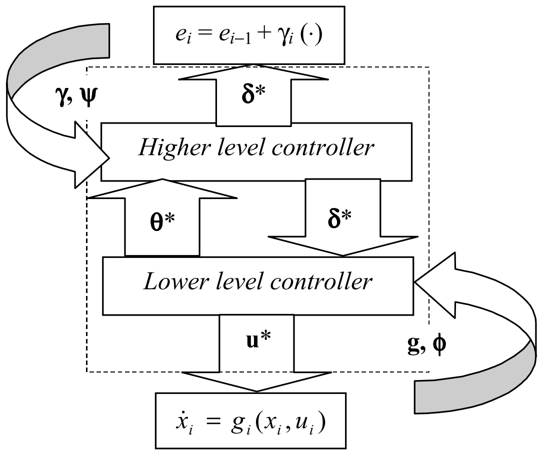

Figure 2. Hybrid controller operation.

Consider now the following example. The linear and decoupled continuous dynamics is given by $\dot{x}_1 = u_1$, $x_1(e_0) = x_0$, $\dot{x}_2 = \alpha x_2 + u_2$, $x_2(e_1) = x_1$, where the event-driven dynamics have the simple form: $e_1 = e_0 + \delta_1(x_1, u_1)$, $e_2 = e_1 + \delta_2(x_2, u_2)$. The cost to minimize is

$$J = \sum_{i=1}^{2} [\phi_i(x_i, u_i, \delta_i)] + \psi_2(e_2), \tag{12}$$

where

$$\phi_1(x_1, u_1, \delta_1) = \int_0^{\delta_1} 0.5 r_1 u_1^2(t) dt,$$

$$\phi_2(x_2, u_2, \delta_2) = 0.5h(x_2^F - x_d^F)^2 + \int_0^{\delta_1} 0.5 r_2 u_2^2(t) dt, \; \psi_2(e_2) = \beta e_2^2. \tag{13}$$

For $\delta_1 = \delta_2 = 0$, it results $x_1^F = x_0$, $x_2^F = x_1^F$ respectively. For a numerical example, setting $r_1 = 2$, $r_2 = 10$, $h = 10$, $x_d = 10$, $x_0 = 0$, $\alpha = 1$ and $\beta = 10$, the following solution of the optimization problem is obtained: it is optimal to start operating in the first mode with $u_1(t) = 5.72$ and *switch* to the second mode at time $e_1 = 0.4$, when $x_1^F = 2.29$. The system operates in the second mode with control $u_2(t) = 1.66e^{-t}$, until time $e_2 = 1.64$, when $x_2^F = 9.67$.

4 CONCLUSIONS

Two optimization problems have been briefly presented, for two distinct classes of hybrid systems. The first one, attached to linear hybrid automata with linear continuous dynamics, is subject to future research. The second one, defined within a dynamic optimization problem with separable cost structure, allows the design of an optimal hybrid controller (in fact, a variable structure system), which defines an optimal switching control policy. It must be emphasized that hybrid automata generally model closed loop systems, i.e. plants coupled to their switching control policy, as presented in the example (see Figure 1). Starting from this remark, it's interesting to investigate a conversion method from the second class of hybrid systems to hybrid automata, in order to study the time-optimal control problem (3).

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